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# ON A COMPARISION OF FOUR ESTIMATES OF A COMMON MEAN BY MULTIPLE CRITERIA DECISION MAKING METHOD

Montip Tiensuwan
*Department of Mathematics, Faculty of science*
*Mahidol University, Rami 6 Road, Bangkok, 10400 Thailand*
*Email: scmts@mahidol.ac.th*

Satinee Lertprapai
*Department of Mathematics, Faculty of science*
*Mahidol University, Rami 6 Road, Bangkok, 10400 Thailand*
*Email: satineel@yahoo.com*

Bimal K. Sinha
*Department of Mathematics and Statistics*
*University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, USA*
*Email: sinha@umbc.edu*

summary

In this paper we consider the problem of estimating the common mean $\mu$ of two independent normal populations with unknown and possibly unequal variances. The purpose of this paper is to compare four unbiased estimates of $\mu$ based on Multiple Criteria Decision Making (MCDM) procedure in order to rank the estimates from the best to the worst in terms of their variances. A simulation study is also used to compare the competing estimates in small samples.

*Keywords and phrases:* Common mean, multiple criteria decision making

*AMS Classification:* 62C25, 62G05

## 1 Introduction

One of the interesting problems in statistical inference is the estimation of a common mean of two independent normal populations with unknown and unequal variances. It has been extensively discussed by Graybill and Deal [2], Sinha [7], and Pal and Sinha [5]. The objective of this paper is to compare various estimates of a common mean based on Multiple Criteria Decision Making (MCDM) which has been advocated by Hwang and Yoon [3], Zeleny [10], and Yoon and Hwang [9]. A lot of research work has been concerned with MCDM problem such as Filar *et al.* [1] and Maitra *et al.* [4]. The MCDM method is briefly described in Section 2 and Section 3 contains the main results for our problem.

## 2 A brief description of MCDM procedure

In the context of a '*discrete*' data matrix $X = (x_{ij}) : K \times N$ where $x'_{ij}s$ represent '*risk*' of $i$th '*source*' for $j$th '*category*', and we need to compare the $K$ rows simultaneously with respect to all the $N$ columns, MCDM is a novel statistical procedure to integrate the multiple indicators $(x_{i1}, \ldots, x_{iN})$ for row $i$ across all indicators into a single meaningful and overall index. This is done by defining an Ideal Row (*IDR*) with the smallest observed value for each column as

$$IDR \quad = \quad (\min_i x_{i1},\ \ldots,\ \min_i x_{iN}) \quad = \quad (u_1\,,\ \ldots\ , u_N)$$

and a Negative-ideal Row *(NIDR)* with the largest observed value for each column as

$$NIDR \quad = \quad (\max_i x_{i1},\ \ldots,\ \max_i x_{iN}) \quad = \quad (v_1\,,\ \ldots\ , v_N).$$

For any given row$i$,we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under $L_1$-norm, we compute

$$L_1(\,i\,, IDR) \quad = \quad \sum_{j\,=\,1}^{N} [x_{ij} - u_j]\ \ w_j$$

$$L_1(\,i\,, NIDR) \quad = \quad \sum_{j\,=\,1}^{N} [\,v_j - x_{ij}]\ \ w_j$$

where $w'_j s$ are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) \quad = \quad \frac{L_1(i\,,\,IDR)}{L_1(i\,,\,IDR)\ +\ L_1(i\,,\,NIDR)}, \quad i\ =\ 1,\ \ldots\ , K. \tag{2.1}$$

Similarly, under $L_2$-norm, we compute

$$L_2(\,i\,, IDR) \quad = \quad [\sum_{j\,=\,1}^{N} (x_{ij} - u_j)^2 w_j]^{1/2}$$

$$L_2(\,i\,, NIDR) \quad = \quad [\sum_{j\,=\,1}^{N} (x_{ij} - v_j)^2 w_j]^{1/2}$$

and compare the rows based on

$$L_2(Index\,_i) \quad = \quad \frac{L_2(i\,,\,IDR)}{L_2(i\,,\,IDR)\ +\ L_2(i\,,\,NIDR)}, \quad i\ =\ 1,\ \ldots\ , K. \tag{2.2}$$

A '*continuous*' version of this setup would involve $x'_{ij}s$ where the index $j$ would vary '*continuously*'. In the context of our problem of comparing four unbiased estimates for estimation of a common mean $\mu$ of two normal populations with unknown variances $\sigma_1^2$ and $\sigma_2^2$ (see Section 3 below), obviously$K = 4$, $x'_{ij}s$ are chosen to represent variances of the four estimates for various value of $\tau = \sigma_2^2/\sigma_1^2$. $L_1$-norm and $L_2$-norm would be redefined as

$$L_1(\,i\,, IDR) \quad = \quad \int_0^\infty [x_i(\tau) - u(\tau)]\; w\,(\tau)\, d\tau \tag{2.3}$$

$$L_1(\,i\,, NIDR) \quad = \quad \int_0^\infty [v(\tau) - x_i(\tau)]\; w\,(\tau)\, d\tau \tag{2.4}$$

$$L_2(\,i\,, IDR) \quad = \quad \sqrt{\int_0^\infty (x_i(\tau) - u(\tau))^2\, w\,(\tau)\, d\tau} \tag{2.5}$$

$$L_2(\,i\,, NIDR) \quad = \quad \sqrt{\int_0^\infty (v(\tau) - x_i(\tau))^2\, w\,(\tau)\, d\tau} \tag{2.6}$$

where $u(\tau) = \min_i \{x_i(\tau)\}$ and $v(\tau) = \max_i \{x_i(\tau)\}$.

# 3 Main results

Let $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ be iid observations from $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, respectively. Let $\bar{x}$, $\bar{y}$,$s_1^2$, $s_2^2$ be sample means and sample variances based on sample size $n$. Define $D = \bar{y} - \bar{x}$. The following four unbiased estimates of $\mu$ are quite standard in the literature:

$\hat{\mu}_1 \;=\; \left(\frac{\bar{x}}{s_1^2} + \frac{\bar{y}}{s_2^2}\right) \Big/ \left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right)$ (Graybill and Deal [2]),

$\hat{\mu}_2 \;=\; \bar{x} + D\left(\frac{s_1^2 + D^2}{s_1^2 + s_2^2 + D^2}\right)$ (Sinha and Mouqadem [8]),

$\hat{\mu}_3 \;=\; \bar{x} + D\min\left\{\frac{s_1^2}{s_1^2 + s_2^2}, \frac{s_2^2}{s_1^2 + s_2^2}\right\}$(Sinha [6]),

$\hat{\mu}_4 \;=\; \bar{x} + D\left(\frac{s_1}{s_1 + s_2}\right)$ (Sinha and Mouqadem [8]).

To compare the above four unbiased estimate of $\mu$ based on their variances, it should be noted that exact analytical expression for the variances are quite complicated. We do this by simulation. It should also be noted that, apart from a common factor $\sigma_1^2$, all the variances depend on $\tau = \sigma_2^2/\sigma_1^2$. We have generated 50 sets of values for $\hat{\mu}_1$, $\hat{\mu}_2, \hat{\mu}_3$ and $\hat{\mu}_4$,when $n = 5$, 10, 15 with $\tau = 0.2$, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, taking $\mu = 0$and $\sigma_1^2 = 1$,and computed their variances. We report below the simulated values of variance for each estimate in Tables 1-3 for different values of $n$.

Finally, we apply MCDM to these four estimates. Figures 1-3 depict the variances of these estimates.

Write $x_{1k}(\tau) = Var_k(\hat{\mu}_1), \ldots, x_{4k}(\tau) = Var_k(\hat{\mu}_4)$ for interval $k$, $k =$ 1,2,...,9. We observe from the graphs that there are 9 intervals of $\tau$(0.2<0.4<0.6<0.8<1.0<1.2<1.4<1.6<1.8<2.0).

Table 1: Values of variances for $n = 5$

| $\tau$ | $Var(\mu_1)$ | $Var(\mu_2)$ | $Var(\mu_3)$ | $Var(\mu_4)$ |
|---|---|---|---|---|
| 0.2 | 0.05725 | 0.05427 | 0.16221 | 0.07517 |
| 0.4 | 0.04082 | 0.04555 | 0.09103 | 0.04090 |
| 0.6 | 0.10196 | 0.08436 | 0.13951 | 0.09152 |
| 0.8 | 0.10867 | 0.10417 | 0.12325 | 0.10336 |
| 1.0 | 0.11361 | 0.11292 | 0.14471 | 0.10631 |
| 1.2 | 0.07718 | 0.10339 | 0.08236 | 0.06705 |
| 1.4 | 0.09673 | 0.13547 | 0.08218 | 0.08971 |
| 1.6 | 0.14825 | 0.15990 | 0.13636 | 0.13831 |
| 1.8 | 0.15172 | 0.15305 | 0.13705 | 0.12479 |
| 2.0 | 0.13989 | 0.16571 | 0.15138 | 0.13040 |

Table 2: Values of variances for $n = 10$

| $\tau$ | $Var(\mu_1)$ | $Var(\mu_2)$ | $Var(\mu_3)$ | $Var(\mu_4)$ |
|---|---|---|---|---|
| 0.2 | 0.01609 | 0.01584 | 0.06086 | 0.01879 |
| 0.4 | 0.02162 | 0.02216 | 0.04282 | 0.02138 |
| 0.6 | 0.03330 | 0.03186 | 0.05040 | 0.03348 |
| 0.8 | 0.03887 | 0.03745 | 0.05115 | 0.03794 |
| 1.0 | 0.03853 | 0.03909 | 0.04945 | 0.03456 |
| 1.2 | 0.05163 | 0.04965 | 0.05379 | 0.05091 |
| 1.4 | 0.06254 | 0.06406 | 0.06375 | 0.05846 |
| 1.6 | 0.05246 | 0.05547 | 0.05215 | 0.04650 |
| 1.8 | 0.07246 | 0.07301 | 0.07360 | 0.06951 |
| 2.0 | 0.07904 | 0.08147 | 0.07557 | 0.07346 |

Table 3: Values of variances for $n = 15$

| $\tau$ | $Var(\mu_1)$ | $Var(\mu_2)$ | $Var(\mu_3)$ | $Var(\mu_4)$ |
|---|---|---|---|---|
| 0.2 | 0.01092 | 0.01090 | 0.04406 | 0.01322 |
| 0.4 | 0.01738 | 0.01716 | 0.03362 | 0.01790 |
| 0.6 | 0.03029 | 0.03070 | 0.03512 | 0.02972 |
| 0.8 | 0.02946 | 0.02992 | 0.02808 | 0.02837 |
| 1.0 | 0.03455 | 0.03480 | 0.03549 | 0.03186 |
| 1.2 | 0.03462 | 0.03627 | 0.03281 | 0.03221 |
| 1.4 | 0.03666 | 0.04187 | 0.03421 | 0.03528 |
| 1.6 | 0.05323 | 0.05579 | 0.04896 | 0.04914 |
| 1.8 | 0.05029 | 0.05450 | 0.05016 | 0.04833 |
| 2.0 | 0.04629 | 0.05257 | 0.04677 | 0.04682 |

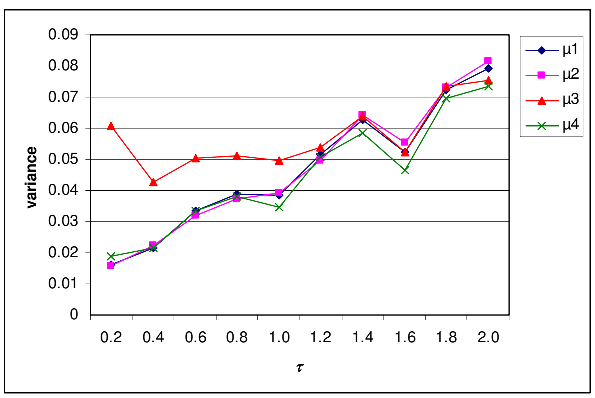


Figure 1: Graphical illustration of variances for four estimates when $n = 5$

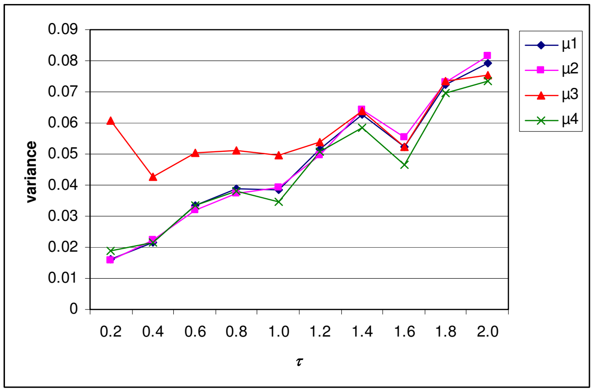


Figure 2: Graphical illustration of variances for four estimates when $n = 10$

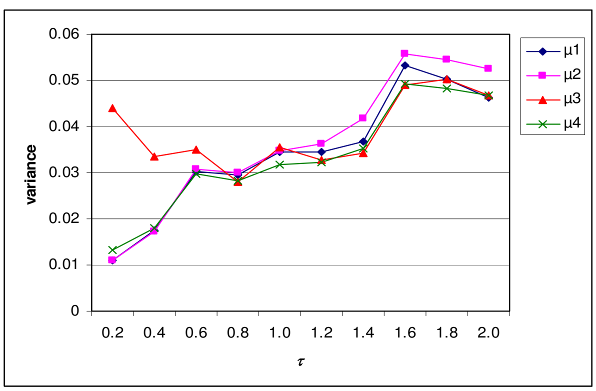


Figure 3: Graphical illustration of variances for four estimates when $n = 15$

Moreover some intervals have intersection point between graphs so we can subdivide these intervals. The main Ideal row and Negative-ideal row are as follows:

$$\begin{aligned}IDR : u(\tau) &= \{u_1(\tau) : 0.2 < \tau < 0.4,\ u_2(\tau) : 0.4 < \tau < 0.6,\ u_3(\tau) : 0.6 < \tau < 0.8,\\ &\quad u_4(\tau) : 0.8 < \tau < 1.0,\ u_5(\tau) : 1.0 < \tau < 1.2,\ u_6(\tau) : 1.2 < \tau < 1.4,\\ &\quad u_7(\tau) : 1.4 < \tau < 1.6,\ u_8(\tau) : 1.6 < \tau < 1.8,\ u_9(\tau) : 1.8 < \tau < 2.0\},\end{aligned}$$

$$\begin{aligned}NIDR : v(\tau) &= \{v_1(\tau) : 0.2 < \tau < 0.4,\ v_2(\tau) : 0.4 < \tau < 0.6,\ v_3(\tau) : 0.6 < \tau < 0.8,\\ &\quad v_4(\tau) : 0.8 < \tau < 1.0,\ v_5(\tau) : 1.0 < \tau < 1.2,\ v_6(\tau) : 1.2 < \tau < 1.4,\\ &\quad v_7(\tau) : 1.4 < \tau < 1.6,\ v_8(\tau) : 1.6 < \tau < 1.8,\ v_9(\tau) : 1.8 < \tau < 2.0\}.\end{aligned}$$

Since we are dealing with a continuous parameter $\tau$,(0.2<0.4<0.6<0.8<1.0<1.2<1.4<1.6< 1.8 < 2.0), a proper formulation of the MCDM procedure can be given as follows.

# 4 Analysis based on $L_1$-norm

For $i$ = 1,2,3,4, we get

$$\begin{aligned}L_1(i, IDR) = {} & \int_{0.2}^{0.4} (x_{i1}(\tau) - u_1(\tau))\ w(\tau)\,d\tau + \int_{0.4}^{0.6} (x_{i2}(\tau) - u_2(\tau))\ w(\tau)\,d\tau\\ & + \int_{0.6}^{0.8} (x_{i3}(\tau) - u_3(\tau))\ w(\tau)\,d\tau + \int_{0.8}^{1.0} (x_{i4}(\tau) - u_4(\tau))\ w(\tau)\,d\tau\\ & + \int_{1.0}^{1.2} (x_{i5}(\tau) - u_5(\tau))\ w(\tau)\,d\tau + \int_{1.2}^{1.4} (x_{i6}(\tau) - u_6(\tau))\ w(\tau)\,d\tau\\ & + \int_{1.4}^{1.6} (x_{i7}(\tau) - u_7(\tau))\ w(\tau)\,d\tau + \int_{1.6}^{1.8} (x_{i8}(\tau) - u_8(\tau))\ w(\tau)\,d\tau\\ & + \int_{1.8}^{2.0} (x_{i9}(\tau) - u_9(\tau))\ w(\tau)\,d\tau\end{aligned}$$

$$\begin{aligned}L_1(i, NIDR) = {} & \int_{0.2}^{0.4} (v_1(\tau) - x_{i1}(\tau))\ w(\tau)\,d\tau + \int_{0.4}^{0.6} (v_2(\tau) - x_{i2}(\tau))\ w(\tau)\,d\tau\\ & + \int_{0.6}^{0.8} (v_3(\tau) - x_{i3}(\tau))\ w(\tau)\,d\tau + \int_{0.8}^{1.0} (v_4(\tau) - x_{i4}(\tau))\, w(\tau)\,d\tau\\ & + \int_{1.0}^{1.2} (v_5(\tau) - x_{i5}(\tau))\ w(\tau)\,d\tau + \int_{1.2}^{1.4} (v_6(\tau) - x_{i6}(\tau))\ w(\tau)\,d\tau\\ & + \int_{1.4}^{1.6} (v_7(\tau) - x_{i7}(\tau))\ w(\tau)\,d\tau + \int_{1.6}^{1.8} (v_8(\tau) - x_{i8}(\tau))\ w(\tau)\,d\tau\\ & + \int_{1.8}^{2.0} (v_9(\tau) - x_{i9}(\tau))\ w(\tau)\,d\tau\end{aligned}$$

The overall index can then be computed. It is clear that for the purpose of comparison of the four estimates, we can work with

$$L_1(Index_i) = \frac{L_1(i\,,IDR)}{L_1(i\,,IDR)+L_1(i\,,NIDR)}, \quad i = 1,2,3,4.$$

## 5 Analysis based on $L_2$-norm

For $i$ = 1,2,3,4, we get

$$L_2(i, IDR) = \sqrt{\begin{array}{l} \int_{0.2}^{0.4} (x_{i1}(\tau) - u_1(\tau))^2 \, w(\tau)\, d\tau + \int_{0.4}^{0.6} (x_{i2}(\tau) - u_2(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{0.6}^{0.8} (x_{i3}(\tau) - u_3(\tau))^2 \, w(\tau)\, d\tau + \int_{0.8}^{1.0} (x_{i4}(\tau) - u_4(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.0}^{1.2} (x_{i5}(\tau) - u_5(\tau))^2 \, w(\tau)\, d\tau + \int_{1.2}^{1.4} (x_{i6}(\tau) - u_6(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.4}^{1.6} (x_{i7}(\tau) - u_7(\tau))^2 \, w(\tau)\, d\tau + \int_{1.6}^{1.8} (x_{i8}(\tau) - u_8(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.8}^{2.0} (x_{i9}(\tau) - u_9(\tau))^2 \, w(\tau)\, d\tau \end{array}}$$

$$L_2(i, NIDR) = \sqrt{\begin{array}{l} \int_{0.2}^{0.4} (v_1(\tau) - x_{i1}(\tau))^2 \, w(\tau)\, d\tau + \int_{0.4}^{0.6} (v_2(\tau) - x_{i2}(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{0.6}^{0.8} (v_3(\tau) - x_{i3}(\tau))^2 \, w(\tau)\, d\tau + \int_{0.8}^{1.0} (v_4(\tau) - x_{i4}(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.0}^{1.2} (v_5(\tau) - x_{i5}(\tau))^2 \, w(\tau)\, d\tau + \int_{1.2}^{1.4} (v_6(\tau) - x_{i6}(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.4}^{1.6} (v_7(\tau) - x_{i7}(\tau))^2 \, w(\tau)\, d\tau + \int_{1.6}^{1.8} (v_8(\tau) - x_{i8}(\tau))^2 \, w(\tau)\, d\tau \\ \quad + \int_{1.8}^{2.0} (v_9(\tau) - x_{i9}(\tau))^2 \, w(\tau)\, d\tau \end{array}}$$

Under $L_2$-norm also, the overall index can be computed.

## 6 Choice of weight functions

Our first weight function $w_1(\tau)$ is defined by $w_1(\tau) = 1$ for every interval. Following Filar *et al.*[1], we also consider two additional choices of $w(\tau)$. The first one, denoted by $w_2(\tau)$, is based on the notion of entropy among $x_{1k}(\tau)$, $x_{2k}(\tau)$, $x_{3k}(\tau)$ and $x_{4k}(\tau)$ for interval $k$ with various values of $\tau$, denoted as $w_{2k}(\tau)$,when $k$ = 1,2,. . . ,9, and the second one, denoted by $w_3(\tau)$, is based on the coefficient of variation of $x_{1k}(\tau)$, $x_{2k}(\tau)$, $x_{3k}(\tau)$ and $x_{4k}(\tau)$ for interval $k$ with various values of $\tau$, denoted as $w_{3k}(\tau)$,when $k$ = 1,2,. . . ,9. It turns out that

$$w_{2k}(\tau) = \frac{1 - \phi_k(\tau)}{\int_0^\infty [\,1 - \phi_k(\tau)\,]\, d\tau}$$

where

$$\phi_k(\tau) \quad = \quad -\frac{1}{\log 4}\sum_{i=1}^{4}\left\{\frac{x_{ik}(\tau)}{\sum\limits_{i=1}^{4} x_{ik}(\tau)} \, . \log\left[\frac{x_{ik}(\tau)}{\sum\limits_{i=1}^{4} x_{ik}(\tau)}\right]\right\}, \; k = 1, 2, ..., 9$$

and

$$w_{3k}(\tau) \quad = \quad \frac{\sqrt{\frac{1}{4}\sum\limits_{i=1}^{4}\left(x_{ik}(\tau) - \bar{x}_k(\tau)\right)^2}}{\frac{1}{4}\sum\limits_{i=1}^{4} x_{ik}(\tau)}$$

therefore,

$$w_3(\tau) \;=\; \frac{\sqrt{3x_{1k}^2(\tau) + 3x_{2k}^2(\tau) + 3x_{3k}^2(\tau) + 3x_{4k}^2(\tau) - 2x_{3k}(\tau)x_{4k}(\tau) - 2x_{2k}(\tau)(x_{3k}(\tau) + x_{4k}(\tau))}}{\sum\limits_{i=1}^{4} x_{ik}(\tau)}$$

$$k = 1, 2, ..., 9.$$

# 7 Comparison of estimates

We report the ranks of the four estimates when compared on the basis of the weight functions $w_1(\tau), w_2(\tau)$and $w_3(\tau)$ using $L_1$-norm and $L_2$-norm in Tables 4-6.

# 8 Conclusion

Based on the above analysis under $L_1$- and $L_2$- norms, we conclude that our preference is uniformly for Sinha and Mouqadem [9] estimate $\hat{\mu}_4$ under three weights $w_1(\tau)$, $w_2(\tau)$ and $w_3(\tau)$. The familiar Graybill and Deal [3] estimate $\hat{\mu}_1$ which often holds rank 2 is also a good candidate for estimation of $\mu$.

# 9 Acknowledgements

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# References

[1] Filar, J.A., Ross, N.P. and Wu, M.L. (1999). *Environmental Assessment Based on Multiple Indicators.* Technical Report: Department of Applied Mathematics, University of South Australia.

Table 4: Analysis based on $L_1$-norm and $L_2$-norm using $w_1(\tau)$

| $n$ | $\mu$ | $L_1$ $(IDR)$ | $L_1$ $(NIDR)$ | $I$ | Rank | $L_2$ $(IDR)$ | $L_2$ $(NIDR)$ | $I$ | Rank |
|---|---|---|---|---|---|---|---|---|---|
| 5 | $\hat{\mu}_1$ | 0.01877 | 0.05409 | 0.25764 | 2 | 0.01634 | 0.04907 | 0.24982 | 2 |
| | $\hat{\mu}_2$ | 0.03303 | 0.03983 | 0.45334 | 3 | 0.03336 | 0.04711 | 0.41453 | 3 |
| | $\hat{\mu}_3$ | 0.04992 | 0.02294 | 0.68515 | 4 | 0.05004 | 0.02721 | 0.64778 | 4 |
| | $\hat{\mu}_4$ | 0.00422 | 0.06864 | 0.05786 | 1 | 0.00606 | 0.05443 | 0.10024 | 1 |
| 10 | $\hat{\mu}_1$ | 0.00464 | 0.01845 | 0.20105 | 2 | 0.00420 | 0.01959 | 0.17640 | 2 |
| | $\hat{\mu}_2$ | 0.00513 | 0.01797 | 0.22209 | 3 | 0.00548 | 0.01991 | 0.21600 | 3 |
| | $\hat{\mu}_3$ | 0.02191 | 0.00118 | 0.94876 | 4 | 0.02078 | 0.00188 | 0.91694 | 4 |
| | $\hat{\mu}_4$ | 0.00062 | 0.02248 | 0.02691 | 1 | 0.00095 | 0.02039 | 0.04431 | 1 |
| 15 | $\hat{\mu}_1$ | 0.00300 | 0.01092 | 0.21542 | 2 | 0.00277 | 0.01289 | 0.17680 | 2 |
| | $\hat{\mu}_2$ | 0.00653 | 0.00739 | 0.46909 | 3 | 0.00604 | 0.01237 | 0.32833 | 3 |
| | $\hat{\mu}_3$ | 0.00876 | 0.00517 | 0.62881 | 4 | 0.01253 | 0.00542 | 0.69790 | 4 |
| | $\hat{\mu}_4$ | 0.00055 | 0.01338 | 0.03940 | 1 | 0.00082 | 0.01311 | 0.05881 | 1 |

Table 5: Analysis based on $L_1$-norm and $L_2$-norm using $w_2(\tau)$

| $n$ | $\mu$ | $L_1$ $(IDR)$ | $L_1$ $(NIDR)$ | $I$ | Rank | $L_2$ $(IDR)$ | $L_2$ $(NIDR)$ | $I$ | Rank |
|---|---|---|---|---|---|---|---|---|---|
| 5 | $\hat{\mu}_1$ | 0.20022 | 0.74575 | 0.21165 | 2 | 0.05103 | 0.18607 | 0.21521 | 2 |
| | $\hat{\mu}_2$ | 0.42675 | 0.51921 | 0.45113 | 3 | 0.11840 | 0.17252 | 0.40698 | 3 |
| | $\hat{\mu}_3$ | 0.67777 | 0.26819 | 0.71649 | 4 | 0.18602 | 0.09312 | 0.66639 | 4 |
| | $\hat{\mu}_4$ | 0.04645 | 0.89951 | 0.04911 | 1 | 0.02082 | 0.20015 | 0.09421 | 1 |
| 10 | $\hat{\mu}_1$ | 0.10033 | 0.32733 | 0.23460 | 2 | 0.01967 | 0.08112 | 0.19512 | 2 |
| | $\hat{\mu}_2$ | 0.11737 | 0.31029 | 0.27444 | 3 | 0.02586 | 0.08182 | 0.24016 | 3 |
| | $\hat{\mu}_3$ | 0.40465 | 0.02301 | 0.94619 | 4 | 0.08645 | 0.00816 | 0.91371 | 4 |
| | $\hat{\mu}_4$ | 0.00889 | 0.41877 | 0.02078 | 1 | 0.00364 | 0.08553 | 0.04081 | 1 |
| 15 | $\hat{\mu}_1$ | 0.04305 | 0.12945 | 0.24956 | 2 | 0.01071 | 0.03920 | 0.21462 | 2 |
| | $\hat{\mu}_2$ | 0.08903 | 0.08346 | 0.51614 | 3 | 0.02167 | 0.03674 | 0.37101 | 3 |
| | $\hat{\mu}_3$ | 0.10431 | 0.06819 | 0.60469 | 4 | 0.03765 | 0.01953 | 0.65839 | 4 |
| | $\hat{\mu}_4$ | 0.00512 | 0.16737 | 0.02970 | 1 | 0.00223 | 0.04126 | 0.05125 | 1 |

Table 6: Analysis based on $L_1$-norm and $L_2$-norm using $w_3(\tau)$

| $n$ | $\mu$ | $L_1$ $(IDR)$ | $L_1$ $(NIDR)$ | $I$ | Rank | $L_2$ $(IDR)$ | $L_2$ $(NIDR)$ | $I$ | Rank |
|---|---|---|---|---|---|---|---|---|---|
| 5 | $\hat{\mu}_1$ | 0.03310 | 0.06225 | 0.34712 | 2 | 0.02244 | 0.04562 | 0.32969 | 2 |
| | $\hat{\mu}_2$ | 0.06630 | 0.02905 | 0.69535 | 4 | 0.04785 | 0.03714 | 0.56302 | 4 |
| | $\hat{\mu}_3$ | 0.04790 | 0.04745 | 0.50238 | 3 | 0.04250 | 0.03899 | 0.52150 | 3 |
| | $\hat{\mu}_4$ | 0.00430 | 0.09105 | 0.04505 | 1 | 0.00542 | 0.05816 | 0.08525 | 1 |
| 10 | $\hat{\mu}_1$ | 0.00271 | 0.00841 | 0.24396 | 2 | 0.00328 | 0.01354 | 0.19497 | 2 |
| | $\hat{\mu}_2$ | 0.00318 | 0.00795 | 0.28537 | 3 | 0.00436 | 0.01371 | 0.24146 | 3 |
| | $\hat{\mu}_3$ | 0.01034 | 0.00079 | 0.92871 | 4 | 0.01437 | 0.00156 | 0.90211 | 4 |
| | $\hat{\mu}_4$ | 0.00029 | 0.01084 | 0.02631 | 1 | 0.00067 | 0.01415 | 0.04543 | 1 |
| 15 | $\hat{\mu}_1$ | 0.00079 | 0.00424 | 0.15642 | 2 | 0.00146 | 0.00899 | 0.13957 | 2 |
| | $\hat{\mu}_2$ | 0.00176 | 0.00327 | 0.35039 | 3 | 0.00322 | 0.00880 | 0.26770 | 3 |
| | $\hat{\mu}_3$ | 0.00359 | 0.00144 | 0.71387 | 4 | 0.00885 | 0.00290 | 0.75333 | 4 |
| | $\hat{\mu}_4$ | 0.00023 | 0.00480 | 0.04543 | 1 | 0.00058 | 0.00885 | 0.06110 | 1 |

Table 7: Summary of rank of four estimates using $w_1(\tau), w_2(\tau)$ and $w_3(\tau)$

| $n$ | $\mu$ | $L_1$-norm | | | $L_2$-norm | | |
|---|---|---|---|---|---|---|---|
| | | $w_1$ | $w_2$ | $w_3$ | $w_1$ | $w_2$ | $w_3$ |
| 5 | $\hat{\mu}_1$ | 2 | 2 | 2 | 2 | 2 | 2 |
| | $\hat{\mu}_2$ | 3 | 3 | 4 | 3 | 3 | 4 |
| | $\hat{\mu}_3$ | 4 | 4 | 3 | 4 | 4 | 3 |
| | $\hat{\mu}_4$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | $\hat{\mu}_1$ | 2 | 2 | 2 | 2 | 2 | 2 |
| | $\hat{\mu}_2$ | 3 | 3 | 3 | 3 | 3 | 3 |
| | $\hat{\mu}_3$ | 4 | 4 | 4 | 4 | 4 | 4 |
| | $\hat{\mu}_4$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | $\hat{\mu}_1$ | 2 | 2 | 2 | 2 | 2 | 2 |
| | $\hat{\mu}_2$ | 3 | 3 | 3 | 3 | 3 | 3 |
| | $\hat{\mu}_3$ | 4 | 4 | 4 | 4 | 4 | 4 |
| | $\hat{\mu}_4$ | 1 | 1 | 1 | 1 | 1 | 1 |

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