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SOME TEST ESTIMATORS FOR THE SCALE PARAMETER OF CLASSICAL PARETO DISTRIBUTION

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summary

In this paper, some test estimators for the scale parameter of a classical Pareto distribution are considered when a prior point guess value of the shape parameter is available. The comparisons of the proposed testimators with the unbiased estimator have been made under the squared error loss function.

Keywords and phrases: Pre - test estimator, Mean square error, Shrinkage factor, Shrinkage testimator.

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1 Introduction

The Pareto distribution has been used to explain many socio-economic phenomenon including assets of firms, insurance claims, city population sizes, occurrence of natural resources, stock price fluctuations, and personal incomes (see Steindl (1965) and Arnold (1983)). The distribution is also a potential model for life testing problems (see Davis and Feldstein (1979), Abdel - Ghaly et al. (1998)).

There are two basic techniques reported in literature to incorporate a prior information about the parameter itself or other and a test of significance while constructing an estimator of the parameter. David and Arnold (1970) obtained an efficient preliminary test estimator for the variance of a normal population when the mean is unknown. Singh et al (1996) proposed shrinkage estimators for the scale parameter of a classical Pareto distribution by shrinking the unbiased estimator towards the guess value. Srivastava and Singh (1982) considered the problem of estimation for the variance of a Normal population utilizing two guessed values of the mean. Singh et al. (1999) have considered an estimator for a scale parameter of Pareto population when a prior point guess value of the shape is available. Recently Singh et al. (2005) has considered some shrinkage and preliminary test estimators for scale of classical Pareto distribution when a point guess value about scale parameter is available. The recent book by Saleh (2006) deserves special attention in this context.

In the present paper, we proposed some pre - test estimators and shrinkage test estimators for scale parameter utilizing the knowledge of a prior point guess value of the shape parameter. Properties of these estimators in terms of the mean square error have been given.

Let y_1,y_2,−−−,y_n be a random sample of size n drawn from a classical Pareto distribution having p. d. f.

$$f(y;\sigma,\theta) = \theta\sigma^{\theta}y^{-(\theta+1)}; y \geq \sigma, \theta > 0. \tag{1.1}$$

The maximum likelihood estimators of shape θ and scale σ are respectively:

$$\hat{\theta} = \left[\frac{1}{n}\sum_{i=1}^{n} log\left(\frac{y_i}{y_{(1)}}\right)\right]^{-1} \quad \text{and} \quad \hat{\sigma} = y_{(1)} = min\left(y_1, y_2, ---, y_n\right)$$

and the unbiased estimates of θ and σ are respectively:

$$\hat{\theta}_u = \frac{(n-2)}{n}\hat{\theta} \quad \text{and} \quad \hat{\sigma}_u = \left[1 - \frac{1}{(n-1)\,\hat{\theta}}\right]\hat{\sigma}\ .$$

The minimum mean square error (MMSE) estimator for σ in the the class of estimators of the form

$$T = l\hat{\sigma}_u\ ,\ l \in R^{+}$$

is

$$\hat{T} = \hat{l}\hat{\sigma}_u\ ,$$

where

$$\hat{l} = \frac{\hat{\theta}_u\,(n-1)\left(n\hat{\theta}_u - 2\right)}{1 + \hat{\theta}_u\,(n-1)\left(n\hat{\theta}_u - 2\right)}\ .$$

The MSE of $\hat{T}$ is given by

$$MSE\left(\hat{T}\right) = \sigma^2\{G\left(0,\infty,w_1\right) + 1\}\ , \tag{1.2}$$

where $G\left(p,q,r\right) = \frac{1}{\Gamma(n-1)}\int_p^q (r).e^{-x}x^{n-2}dx$, $d_1 = \dfrac{n\delta\theta_0}{(n\delta\theta_0 - 2)}$, $d_2 = \dfrac{2n\delta\theta_0}{(n\delta\theta_0 - 1)}$, $\delta = \dfrac{\theta}{\theta_0}$, $w_1 = \left(d_1 f_1^2 - d_2 f_1\right)$, $f_1 = \hat{l}\left(1 - \dfrac{x}{n\delta\,(n-1)\,\theta_0}\right)$ and r being a function of x .

Similarly, if we assume that $\theta = \theta_0$ is known, the unbiased estimator is $\widetilde{\sigma}_u = \left[1 - \frac{1}{n\theta_0}\right] \hat{\sigma}$ and the the MMSE estimator for σ in the class $c\,\widetilde{\sigma}_u$ is $T^* = \hat{c}\,\widetilde{\sigma}_u$, where $\hat{c} = n\,\theta_0 \dfrac{(n\,\theta_0 - 2)}{(n\,\theta_0 - 1)^2}$ with

$$MSE\,(T^*) = \sigma^2\,(n\theta_0 - 1)^{-2}\ . \tag{1.3}$$

2 Pre - Test Estimators and Its Properties

When a prior point guess value θ_0 of shape θ is available, the pre $-$ test estimator for σ incorporating the preliminary test of significance $H_0\ : \theta = \theta_0$ against $H_1\ : \theta \neq \theta_0$ is then defined as follows

$$P_1 = \begin{cases} \hat{c}\tilde{\sigma}_u & if\ \ \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{\sigma_u} & else \end{cases},$$

where r_1 and r_2 are such that $P\left[\chi^2_{2(n-1)} > r_2\right] = \dfrac{\alpha}{2}$ and $P\left[\chi^2_{2(n-1)} \leqslant r_1\right] = \dfrac{\alpha}{2}$, $\chi^2_{2(n-1)}$ is a chi $-$ square variate with $2\,(n-1)$ degrees of freedom and α is a preassigned level of significance.

For $\hat{c} = 1, P_1$ reduces to the pre $-$ test estimator as considered by Singh et al. (1999). The expression for the mean square error of P_1 is

$$\begin{aligned} MSE(P_1) = \sigma^2 \{ & G\,(0, \infty, w_2) - G\,(x_1, x_2, w_2) \\ & + a\ G\,(x_1, x_2, w_0) + 1\}\ ; \end{aligned} \tag{2.1}$$

where $a = \frac{n\delta\theta_0(n\theta_0-2)(2(n\theta_0-1)-n\theta_0(n\delta\theta_0-1))}{(n\theta_0-1)^2(n\delta\theta_0-2)(n\delta\theta_0-1)}$, $w_2 = \left(d_1 f_2^2 - d_2 f_2\right)$, $w_0 = 1$, $f_2 = \left(1 - \frac{x}{n\delta\theta_0(n-1)}\right)$, $x_1 = \frac{\delta r_1}{2}$ and $x_2 = \frac{\delta r_2}{2}$.

The expression of relative bias is

$$RB\,(P_1) = \frac{d_2}{2} G\,(x_1, x_2, u_1)\ \ ; \tag{2.2}$$

where $u_1 = \left(\frac{n\theta_0 - 2}{n\theta_0 - 1} - f_2\right)$.

Two other combinations for pre $-$ test estimators of σ, are considered as

$$P_2 = \begin{cases} \tilde{\sigma}_u & if\ \ \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{l}\hat{\sigma_u} & else \end{cases}$$

and

$$P_3 = \begin{cases} \hat{c}\tilde{\sigma}_u & if\ \ \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{l}\hat{\sigma_u} & else \end{cases}.$$

The mean square errors of these estimators are

$$MSE(P_2) = \sigma^2\{G(0,\infty,w_1) - G(x_1,x_2,w_1) + a_1\,G(x_1,x_2,w_0) + 1\}\;; a_1 = \frac{n\delta\theta_0\,(3 - n\delta\theta_0)}{(n\delta\theta_0 - 2)\,(n\delta\theta_0 - 1)}$$

and

$$MSE(P_3) = \sigma^2\{G(0,\infty,w_1) - G(x_1,x_2,w_1) + a\,G(x_1,x_2,w_0) + 1\}.$$

The relative biases are

$$RB\,(P_2) = \frac{d_2}{2}\left\{G(0,\infty,f_1) + G(x_1,x_2,u_2) - \frac{2}{d_2}\right\}$$

and

$$RB\,(P_3) = \frac{d_2}{2}\left\{G(0,\infty,f_1) + G(x_1,x_2,u_3) - \frac{2}{d_2}\right\}\;;$$

where $u_2 = \left(\frac{n\theta_0 - 1}{n\theta_0} - f_1\right)$ and $u_3 = \left(\frac{n\theta_0 - 2}{n\theta_0 - 1} - f_1\right)$.

Also, the mean square error for $\hat{\sigma_u}$ is

$$MSE\,(\hat{\sigma}_u) = \sigma^2\,(\delta\theta_0\,(n-1)\,(n\delta\theta_0 - 2))^{-1}\,.$$

The relative efficiencies of $P_i; i = 1,2,3$ with respect to $\hat{\sigma}_u$ is defined as

$$RE\,(P_i,\hat{\sigma}_u) = \frac{MSE\,(\hat{\sigma}_u)}{MSE\,(P_i)}\;; i = 1,2,3.$$

The expressions of relative biases and relative efficiencies are the functions of n, δ, θ_0 and α. For selected values of $n = 06, 08, 10, 15; \theta_0 = 0.60(0.40)1.40; \delta = 0.50(0.25)1.50$ and $\alpha = 0.01, 0.05, 0.10, 0.20$, the relative biases (not presented here) and relative efficiencies have been calculated and presented them in Tables $1 - 3$.

The relative biases are negligible and they lie between -0.302 and 0.222.The relative biases $|RB\,(P_2)|$ and $|RB\,(P_3)|$ decreases as n increases. From tables $1 - 3$, we observed that the pre $-$ test estimators P_1 and P_2 performed better than the usual unbiased estimator when $\delta \in (0.50, 1.50]$ for all considered values of n, θ_0 and α. On the other hand, the test estimator P_3 performs better for all considered values of n, θ_0, δ and α. The relative efficiency attains maximum at $\delta = 1$ at which it increases with the values α for pre - test estimator P_1 and P_3 (except at $\theta_0 = 0.60$ in case of P_3).

On other hand, the relative efficiency of P_2 increases as α increases when $\delta \in [1.25, 1.50]$ and

n is small. The values of relative efficiency decreases as n increases (except at $\delta = 0.50$ for P_1 and P_2) . As the gain in relative efficiencies of P_3 is larger than P_1 and P_2 when $\delta \in [0.50, 1.00]$ and the gain in relative efficiencies of P_2 is larger than P_1 and P_2 when $\delta \in [1.25, 1.50]$. One would prefer the estimator P_3 for smaller δ and P_2 for larger δ.

3 Shrinkage Test Estimators and Their Properties

Thompson (1968) investigated an estimator that performs better than a usual estimator when a prior point guess value is in neighborhood of the true value of the parameter. This technique has been used by many authers like Mehta and Srinivasan (1971), Pandey and Singh (1977), Pandey (1979) and others when a guess value of the parameter for which the estimator is proposed, is available. The shrinkage estimator for scale parameter σ is proposed as

$$\widetilde{T} = k\hat{\sigma}_u + (1-k)\,\tilde{\sigma}_u, \tag{3.1}$$

where the constant $k \in [0,1]$ is the shrinkage factor, specified by the experimenter according to his belief in θ_0. The value of k which, minimizes the MSE of $\widetilde{T}$ is given by

$$k = \frac{(\theta-\theta_0)\,(n\,(\theta-\theta_0)-1)\,/\,(n\theta-1)}{\left(\theta\,(n-1)\,(\theta-2\theta_0)-n\theta_0^2\right)/\theta\,(n-1)}.$$

The value of k depends on unknown parameter θ. Replacing θ by its unbiased estimate $\hat{\theta}_u$, we obtain an estimate $\hat{k}$ for k and then the improved shrinkage estimator is given by

$$\widetilde{\widetilde{T}} = \hat{k}\hat{\sigma}_u + \left(1-\hat{k}\right)\tilde{\sigma}_u \tag{3.2}$$

with mean square error

$$MSE\left(\widetilde{\widetilde{T}}\right) = \sigma^2\left\{G\,(0,\infty,w_3)+1\right\};$$

where $w_3 = \left(d_1 f_3^2 - d_2 f_3\right)$ and $f_3 = \hat{k}\left(\frac{\delta(n-1)-x}{n\delta\theta_0(n-1)}\right) + \left(\frac{n\theta_0-1}{n\theta_0}\right)$.

The shrinkage test estimator for testing the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ for a given set of data is defined as

$$P_{S1} = \begin{cases} \hat{k}\,(\hat{\sigma}_u - \widetilde{\sigma}_u) + \widetilde{\sigma}_u & if \;\; \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{l}\hat{\sigma}_u & else \end{cases}$$

The mean square error for P_{S1} is

$$\begin{aligned} MSE(P_{S1}) = \sigma^2\{&G\,(x_1,x_2,w_3) + G\,(0,\infty,w_1) \\ &-G\,(x_1,x_2,w_1) + 1\}\ . \end{aligned} \tag{3.3}$$

The relative bias for P_{S1} is

$$RB(P_{S1}) = \frac{d_2}{2}\left\{G\left(0,\infty,f_1\right) + G\left(x_1,x_2,u_4\right) - \frac{2}{d_2}\right\}; \tag{3.4}$$

where $u_4 = \left(f_3 - f_1\right)$.

The test statistic under H_0, $r_1 \leq \frac{2n\theta_0}{\hat{\theta}} \leq r_2$, may be written as $0 \leq \frac{1}{r_2 - r_1}\left(\frac{2n\theta_0}{\hat{\theta}} - r_1\right) = k_1$ (say) ≤ 1. Therefore, the choice for the shrinkage factor will be k_1 (see Waikar et al. (1984)). Based on this choice, the shrinkage test estimator is given by

$$P_{S2} = \begin{cases} k_1\left(\hat{\sigma}_u - \widetilde{\sigma}_u\right) + \widetilde{\sigma}_u & if\ \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{l}\hat{\sigma}_u & else \end{cases}.$$

When $H_0 : \theta = \theta_0$ is accepted , $r_1 \leq 2(n-1) \leq r_2 \Rightarrow \frac{r_1}{2(n-1)} \leq 1$. If one is interested in taking smaller values of shrinkage factor, he can take $\frac{r_1}{2(n-1)} \cong 1$. Thus the shrinkage test estimator is given by

$$P_{S3} = \begin{cases} k_2\left(\hat{\sigma}_u - \widetilde{\sigma}_u\right) + \widetilde{\sigma}_u & if\ \frac{2n\theta_0}{r_2} \leq \hat{\theta} \leq \frac{2n\theta_0}{r_1} \\ \hat{l}\hat{\sigma}_u & else \end{cases},$$

where $k_2 = \frac{2(n-1)}{r_2 - r_1}\left|\frac{n\theta_0}{(n-1)\hat{\theta}} - 1\right|$; it may be possible that k_2 is negative so we make it positive. Adke et al. (1987) and Pandey et al. (1988) considered this type of shrinkage factor.

The mean square errors and relative biases for these shrinkage test estimators are easily obtained by making some modifications in (3.3) and (3.4) respectively. The relative efficiency of P_{Si} ; $i = 1, 2, 3$. with respect to $\hat{\sigma}_u$ is given by

$$RE\left(P_{Si}, \hat{\sigma}_u\right) = \frac{MSE\left(\hat{\sigma}_u\right)}{MSE\left(P_{Si}\right)} \ ; i = 1, 2, 3.$$

The expressions of relative efficiency and relative bias are the functions of n, α, θ_0 and δ. For the same set of values as considered earlier, the relative biases (not presented here) and relative efficiencies (Table 4 to 6) have been calculated. The relative bias is negligible and lies between 0 to -0.235. The value of $\left|RB\left(P_{Si}\right)\right|$; $i = 1, 2, 3$ decreases as sample size n increase but it increases with level of significance α increases. For P_{S2} and P_{S3}, the relative biases increase when $\delta \in [0.50, 1.50]$, whereas it increases for P_{S1} for all δ except $\alpha = 0.20$.

From the Table 4 to 6, we conclude that the shrinkage test estimators perform better than the usual unbiased estimator for all considered value of n, α, θ_0 and δ. The relative efficiencies attain maximum at $\delta = 1$ and decrease as the level of significance α increases for fixed θ_0 when δ is large. As the sample size n increases the relative efficiencies of these test estimators decrease for

$\delta \in [1.00, 1.50]$. We have following observations, from tabulated value of $RE\left(P_{Si}, \hat{\sigma}_u\right)$; $i = 1, 2, 3$.

1. If a guess value is equal to the parameter, i.e., $\delta = 1$, then the performances of P_{S1} dominates over all other test estimators.
2. The test estimators P_{S1} should also be preferred when $\delta \in [0.50, 0.75]$ and $\alpha \geq 0.05$. On other hand, P_{S2} should be preferred when $\delta \in [1.25, 1.50]$ and for the same value of α.
3. At $\alpha = 0.01$, P_{S3} may be chosen for small $n \leq 8$ when $\delta \in [1.25, 1.50]$.

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Table 1: ► $RE\,(P_1, \hat{\sigma}_u)$

θ_0	δ	$n = 06$				$n = 08$			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	0.8618	0.8289	0.8307	0.8426	0.9484	0.9345	0.9373	0.9437
	0.75	2.3348	2.3071	2.3247	2.3617	2.3320	2.2981	2.3123	2.3585
	1.00	2.7357	2.7541	2.7590	2.7907	2.7330	2.7421	2.7459	2.7722
	1.25	2.4309	2.4637	2.4652	2.4473	2.4294	2.4492	2.4410	2.4138
	1.50	1.4042	1.4165	1.4154	1.3864	1.4034	1.4107	1.4046	1.3720
1.00	0.50	1.0942	1.0469	1.0397	1.0415	1.0955	1.0737	1.0725	1.0753
	0.75	2.4514	2.3975	2.3810	2.3907	2.4144	2.3710	2.3668	2.3841
	1.00	2.7708	2.8205	2.8293	2.8409	2.7622	2.7883	2.7901	2.7995
	1.25	2.4401	2.4993	2.5230	2.5305	2.4376	2.4776	2.4843	2.4724
	1.50	1.4057	1.4274	1.4401	1.4389	1.4047	1.4192	1.4229	1.4085
1.40	0.50	1.1845	1.1349	1.1251	1.1240	1.1563	1.1323	1.1299	1.1314
	0.75	2.5007	2.4577	2.4381	2.4373	2.4515	2.4093	2.4009	2.4105
	1.00	2.7852	2.8510	2.8651	2.8753	2.7748	2.8103	2.8135	2.8195
	1.25	2.4435	2.5128	2.5453	2.5634	2.4408	2.4889	2.5017	2.4964
	1.50	1.4062	1.4311	1.4484	1.4566	1.4052	1.4224	1.4296	1.4219
θ_0	δ	$n = 10$				$n = 15$			
0.60	0.50	0.9888	0.9842	0.9865	0.9898	1.0371	1.0377	1.0383	1.0388
	0.75	2.3228	2.2775	2.2853	2.3451	2.3067	2.2164	2.2316	2.2501
	1.00	2.7286	2.7326	2.7385	2.7656	2.7160	2.7198	2.7326	2.7617
	1.25	2.4277	2.4369	2.4223	2.3911	2.4219	2.4101	2.3870	2.3548
	1.50	1.4028	1.4057	1.3952	1.3585	1.4013	1.3948	1.3755	1.3305
1.00	0.50	1.0962	1.0872	1.0877	1.0897	1.1018	1.1015	1.1020	1.1023
	0.75	2.3844	2.3553	2.3592	2.3785	2.3396	2.3385	2.3480	2.3611
	1.00	2.7538	2.7663	2.7675	2.7799	2.7339	2.7358	2.7425	2.7618
	1.25	2.4355	2.4610	2.4573	2.4361	2.4293	2.4284	2.4110	2.3829
	1.50	1.4040	1.4131	1.4105	1.3874	1.4024	1.4010	1.3872	1.3506
1.40	0.50	1.1415	1.1311	1.1310	1.1325	1.1295	1.1290	1.1293	1.1296
	0.75	2.4131	2.3816	2.3817	2.3956	2.3558	2.3513	2.3588	2.3701
	1.00	2.7649	2.7828	2.7836	2.7919	2.7421	2.7443	2.7492	2.7651
	1.25	2.4386	2.4709	2.4717	2.4550	2.4323	2.4361	2.4211	2.3949
	1.50	1.4045	1.4160	1.4163	1.3983	1.4029	1.4034	1.3918	1.3586

Table 2: ► $RE\,(P_2, \hat{\sigma}_u)$

θ_0	δ	$n = 06$				$n = 08$			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	1.6912	1.0503	0.9020	0.7983	1.0578	0.9172	0.8761	0.8463
	0.75	3.3685	3.0906	2.8354	2.5839	2.7425	2.5167	2.4095	2.3128
	1.00	3.5781	3.4423	3.4352	3.3837	3.0276	3.0695	3.0598	3.0318
	1.25	2.7119	2.8213	2.8923	2.9644	2.5671	2.6491	2.6920	2.7267
	1.50	1.4995	1.5366	1.5725	1.6217	1.4496	1.4775	1.5000	1.5236
1.00	0.50	1.3104	1.1358	1.0750	1.0258	1.1420	1.0643	1.0412	1.0248
	0.75	2.7905	2.6818	2.6000	2.5113	2.5733	2.4711	2.4197	2.3765
	1.00	2.9572	3.0676	3.0967	3.1017	2.8713	2.9319	2.9386	2.9339
	1.25	2.5240	2.6277	2.6933	2.7581	2.4854	2.5622	2.6001	2.6272
	1.50	1.4344	1.4692	1.5016	1.5423	1.4200	1.4459	1.4655	1.4824
1.40	0.50	1.3094	1.1905	1.1481	1.1144	1.1852	1.1256	1.1087	1.0974
	0.75	2.6859	2.6279	2.5749	2.5183	2.5474	2.4742	2.4374	2.4093
	1.00	2.8830	2.9962	3.0296	3.0428	2.8377	2.9010	2.9106	2.9105
	1.25	2.4856	2.5862	2.6486	2.7081	2.4667	2.5409	2.5760	2.5990
	1.50	1.4202	1.4541	1.4849	1.5219	1.4130	1.4380	1.4563	1.4704
θ_0	δ	$n = 10$				$n = 15$			
0.60	0.50	0.9834	0.9231	0.9072	0.8967	0.9868	0.9780	0.9765	0.9758
	0.75	2.5033	2.3536	2.2947	2.2489	2.3102	2.2524	2.2392	2.2332
	1.00	2.9111	2.9298	2.9187	2.9024	2.8029	2.7984	2.7939	2.7961
	1.25	2.5154	2.5781	2.6038	2.6184	2.4703	2.4996	2.5021	2.4956
	1.50	1.4309	1.4523	1.4660	1.4739	1.4144	1.4241	1.4234	1.4087
1.00	0.50	1.0916	1.0542	1.0450	1.0394	1.0736	1.0682	1.0674	1.0671
	0.75	2.4654	2.3818	2.3511	2.3314	2.3426	2.3076	2.3024	2.3018
	1.00	2.8308	2.8624	2.8614	2.8582	2.7776	2.7800	2.7796	2.7864
	1.25	2.4693	2.5269	2.5478	2.5555	2.4516	2.4757	2.4740	2.4620
	1.50	1.4139	1.4335	1.4446	1.4469	1.4077	1.4156	1.4126	1.3937
1.40	0.50	1.1379	1.1085	1.1019	1.0982	1.1100	1.1059	1.1054	1.1053
	0.75	2.4647	2.4003	2.3782	2.3670	2.3582	2.3312	2.3289	2.3116
	1.00	2.8120	2.8461	2.8472	2.8469	2.7711	2.7751	2.7757	2.7837
	1.25	2.4580	2.5131	2.5317	2.5359	2.4464	2.4681	2.4645	2.4499
	1.50	1.4097	1.4285	1.4384	1.4383	1.4059	1.4130	1.4090	1.3883

Table 3: ► $RE\,(P_3, \hat{\sigma}_u)$

θ_0	δ	$n = 06$				$n = 08$			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	2.5043	1.4596	1.2359	1.0832	1.2266	1.0749	1.0801	1.0074
	0.75	3.3930	3.5261	3.5166	3.3438	2.9191	2.7964	2.7328	2.6764
	1.00	3.9240	3.6762	3.5768	3.6046	3.0443	3.1193	3.1350	3.1401
	1.25	2.7068	2.7916	2.8299	2.8376	2.5614	2.6202	2.6362	2.6237
	1.50	1.4977	1.5190	1.5261	1.5020	1.4480	1.4630	1.4644	1.4385
1.00	0.50	1.4169	1.2489	1.1893	1.1409	1.2098	1.1373	1.1159	1.1010
	0.75	2.8583	2.8162	2.7698	2.7186	2.6334	2.5726	2.5409	2.5176
	1.00	2.9623	3.0858	3.1269	3.1495	2.8757	2.9452	2.9587	2.9631
	1.25	2.5215	2.6131	2.6627	2.6951	2.4823	2.5466	2.5697	2.5710
	1.50	1.4336	1.4614	1.4807	1.4875	1.4191	1.4386	1.4475	1.4392
1.40	0.50	1.3663	1.2566	1.2174	1.1865	1.2278	1.1731	1.1579	1.1481
	0.75	2.7213	2.6997	2.6671	2.6329	2.5826	2.5346	2.5101	2.4948
	1.00	2.8852	3.0041	3.0427	3.0636	2.8397	2.9071	2.9198	2.9240
	1.25	2.4839	2.5763	2.6278	2.6653	2.4645	2.5300	2.5550	2.5600
	1.50	1.4197	1.4490	1.4712	1.4860	1.4124	1.4332	1.4442	1.4412

θ_0	δ	$n = 10$				$n = 15$			
0.60	0.50	1.0903	1.0306	1.0449	1.0045	1.0493	1.0215	1.0401	1.0005
	0.75	2.6400	2.5520	2.5185	2.4957	2.4097	2.3763	2.3715	2.3722
	1.00	2.9249	2.9657	2.9698	2.9717	2.8134	2.8196	2.8210	2.8294
	1.25	2.5087	2.5483	2.5497	2.5257	2.4611	2.4669	2.4494	2.4159
	1.50	1.4291	1.4388	1.4344	1.4033	1.4124	1.4108	1.3953	1.3534
1.00	0.50	1.1442	1.1100	1.1018	1.0971	1.1082	1.1036	1.1009	1.0827
	0.75	2.5215	2.4666	2.4484	2.4407	2.3912	2.3691	2.3685	2.3726
	1.00	2.8348	2.8729	2.8763	2.8786	2.7809	2.7868	2.7883	2.7971
	1.25	2.4656	2.5101	2.5174	2.5031	2.4462	2.4566	2.4433	2.4155
	1.50	1.4130	1.4263	1.4277	1.4092	1.4065	1.4081	1.3969	1.3626
1.40	0.50	1.1728	1.1462	1.1404	1.1374	1.1339	1.1303	1.1300	1.1299
	0.75	2.4994	2.4533	2.4394	2.4361	2.3901	2.3718	2.3726	2.3778
	1.00	2.8139	2.8511	2.8542	2.8566	2.7727	2.7784	2.7800	2.7889
	1.25	2.4554	2.5013	2.5103	2.4992	2.4426	2.4546	2.4428	2.4171
	1.50	1.4091	1.4236	1.4269	1.4125	1.4051	1.4079	1.3981	1.3667

Table 4: ► $RE\,(P_{S1}, \hat{\sigma}_u)$

θ_0	δ	n = 06				n = 08			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	1.3477	2.0456	2.9436	2.9520	2.0355	2.6291	2.9147	3.2239
	0.75	2.5335	2.8486	2.9949	3.2899	2.5546	2.7513	2.8735	3.0065
	1.00	3.6363	3.5668	3.5732	3.6215	3.4512	3.4066	3.4167	3.4466
	1.25	3.3285	3.1873	3.1194	3.0407	3.0829	2.9859	2.9438	2.8980
	1.50	1.9839	1.9332	1.8886	1.8335	1.7772	1.7442	1.7191	1.6942
1.00	0.50	1.5848	1.8466	1.9491	2.0819	1.6533	1.7998	1.8477	1.8913
	0.75	2.5799	2.6846	2.7321	2.8234	2.5970	2.6931	2.7488	2.8000
	1.00	3.5755	3.4491	3.4187	3.4136	3.4243	3.3546	3.3473	3.3524
	1.25	3.1680	3.0222	2.9763	2.9216	2.9915	2.8986	2.8568	2.8284
	1.50	1.8081	1.7648	1.7281	1.6858	1.6719	1.6435	1.6232	1.6063
1.40	0.50	1.5482	1.7061	1.7526	1.8132	1.6266	1.7231	1.7500	1.7733
	0.75	2.6275	2.6887	2.7212	2.7787	2.6298	2.7003	2.7399	2.7720
	1.00	3.5650	3.4300	3.3940	3.3802	3.4195	3.3454	3.3347	3.3347
	1.25	3.1146	2.9955	2.9247	2.8630	2.9431	2.8548	2.8164	2.7924
	1.50	1.7498	1.7096	1.6761	1.6389	1.6341	1.6075	1.5892	1.5755
θ_0	δ	n = 10				n = 15			
0.60	0.50	1.9329	2.1473	2.2207	2.2748	1.8271	1.8518	1.8564	1.8587
	0.75	2.6031	2.7625	2.8380	2.9032	2.6913	2.7644	2.7825	2.7908
	1.00	3.3635	3.3428	3.3549	3.3733	3.2746	3.2798	3.2850	3.2824
	1.25	2.9462	2.8953	2.8669	2.8378	2.7959	2.7633	2.7606	2.7575
	1.50	1.6730	1.6497	1.6354	1.6272	1.5521	1.5436	1.5432	1.5275
1.00	0.50	1.7020	1.7779	1.7994	1.8134	1.7281	1.7395	1.7413	1.7420
	0.75	2.6290	2.7176	2.7539	2.7785	2.6891	2.7311	2.7376	2.7372
	1.00	3.3472	3.3107	3.3119	3.3155	3.2658	3.2630	3.2635	3.2556
	1.25	2.8785	2.8121	2.7895	2.7813	2.7432	2.7269	2.7187	2.7116
	1.50	1.5986	1.5786	1.5678	1.5653	1.5098	1.5024	1.5001	1.4923
1.40	0.50	1.6753	1.7270	1.7401	1.7478	1.7093	1.7172	1.7182	1.7185
	0.75	2.6501	2.7184	2.7438	2.7574	2.6941	2.7259	2.7288	2.7255
	1.00	3.3441	3.3045	3.3033	3.3036	3.2640	3.2593	3.2586	3.2493
	1.25	2.8432	2.7806	2.7607	2.7562	2.7226	2.7187	2.7030	2.6991
	1.50	1.5708	1.5522	1.5427	1.5426	1.5132	1.5096	1.4902	1.4866

Table 5: ► $RE\,(P_{S2}, \hat{\sigma}_u)$

θ_0	δ	$n = 06$				$n = 08$			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	1.6371	1.9978	2.5142	2.4818	2.1456	2.6086	2.9033	3.1962
	0.75	2.7694	2.7879	2.9540	3.1507	2.6694	2.7358	2.8060	2.9095
	1.00	3.6031	3.4787	3.4296	3.4125	3.3920	3.3059	3.2813	3.2804
	1.25	3.2947	3.2905	3.2089	3.1337	3.0573	3.0266	2.9784	2.9417
	1.50	2.0150	1.9878	1.9622	1.9199	1.8669	1.8580	1.8459	1.8236
1.00	0.50	1.6855	1.8116	1.9409	2.0714	1.7060	1.7936	1.8431	1.8830
	0.75	2.7001	2.6595	2.7037	2.7499	2.6669	2.6791	2.7018	2.7364
	1.00	3.4868	3.3771	3.3321	3.2994	3.3526	3.2844	3.2614	3.2486
	1.25	3.1017	3.0650	2.9895	2.9219	2.9560	2.9085	2.8824	2.8521
	1.50	1.8622	1.8659	1.8324	1.8076	1.7784	1.7782	1.7741	1.7638
1.40	0.50	1.6285	1.6815	1.7482	1.8048	1.6708	1.7282	1.7460	1.7666
	0.75	2.7121	2.6727	2.6955	2.7178	2.6841	2.6866	2.6986	2.7171
	1.00	3.4575	3.3624	3.3246	3.2937	3.3430	3.2858	3.2661	3.2519
	1.25	3.0380	2.9710	2.9350	2.8920	2.9197	2.8823	2.8631	2.8402
	1.50	1.8103	1.8281	1.7914	1.7717	1.7460	1.7492	1.7482	1.7428
θ_0	δ	$n = 10$				$n = 15$			
0.60	0.50	1.9729	2.1459	2.2129	2.2671	1.8291	1.8504	1.8553	1.8581
	0.75	2.6591	2.7253	2.7764	2.8385	2.6863	2.7315	2.7520	2.7696
	1.00	3.3003	3.2435	3.2427	3.2397	3.2138	3.2037	3.1960	3.1948
	1.25	2.9737	2.8996	2.8713	2.8556	2.8289	2.8021	2.7885	2.7754
	1.50	1.7989	1.7988	1.7936	1.7814	1.7309	1.7404	1.7423	1.7403
1.00	0.50	1.7258	1.7769	1.7952	1.8093	1.7296	1.7385	1.7404	1.7415
	0.75	2.6660	2.6887	2.7082	2.7314	2.6841	2.7040	2.7126	2.7197
	1.00	3.2871	3.2425	3.2299	3.2244	3.2186	3.2007	3.1969	3.1952
	1.25	2.8966	2.8517	2.8351	2.8160	2.8022	2.7883	2.7806	2.7708
	1.50	1.7385	1.7456	1.7466	1.7433	1.6994	1.7140	1.7200	1.7233
1.40	0.50	1.6959	1.7260	1.7364	1.7443	1.7107	1.7162	1.7174	1.7181
	0.75	2.6801	2.6921	2.7030	2.7156	2.6891	2.7009	2.7057	2.7094
	1.00	3.2852	3.2479	3.2362	3.2279	3.2221	3.2059	3.2007	3.1960
	1.25	2.8735	2.8364	2.8250	2.8107	2.7918	2.7834	2.7784	2.7704
	1.50	1.7155	1.7254	1.7289	1.7293	1.6980	1.7035	1.7112	1.7167

Table 6: ► $RE\,(P_{S3}, \hat{\sigma}_u)$

θ_0	δ	$n = 06$				$n = 08$			
$\alpha \rightarrow$		0.01	0.05	0.10	0.20	0.01	0.05	0.10	0.20
0.60	0.50	1.2425	1.3224	1.9490	2.7330	2.0349	2.6029	2.9099	3.2123
	0.75	2.5915	2.7504	2.9066	3.1358	2.5750	2.7159	2.8168	2.9224
	1.00	3.5619	3.4095	3.3571	3.3221	3.3901	3.2931	3.2675	3.2400
	1.25	3.4341	3.1423	3.0209	2.8758	3.1281	2.9505	2.8693	2.7670
	1.50	2.0414	1.9769	1.9114	1.8111	1.8719	1.8331	1.7905	1.7208
1.00	0.50	1.5923	1.8080	1.9358	2.0741	1.6594	1.7916	1.8449	1.8868
	0.75	2.6155	2.6339	2.6785	2.7400	2.6108	2.6671	2.7089	2.7436
	1.00	3.5166	3.3514	3.2877	3.2319	3.3793	3.2786	3.2468	3.2101
	1.25	3.2014	3.0178	2.9209	2.8068	2.9917	2.8854	2.8221	2.7446
	1.50	1.9040	1.8470	1.8215	1.7513	1.7911	1.7701	1.7422	1.6952
1.40	0.50	1.5606	1.6799	1.7440	1.8064	1.6332	1.7167	1.7475	1.7695
	0.75	2.6555	2.6499	2.6751	2.7086	2.6412	2.6770	2.7046	2.7227
	1.00	3.5111	3.3510	3.2889	3.2325	3.3803	3.2827	3.2512	3.2141
	1.25	3.1257	2.9798	2.8937	2.7933	2.9590	2.8635	2.8079	2.7411
	1.50	1.8567	1.7997	1.7893	1.7327	1.7614	1.7469	1.7247	1.6866
θ_0	δ	$n = 10$				$n = 15$			
0.60	0.50	1.9346	2.1455	2.2185	2.2707	1.8270	1.8515	1.8560	1.8582
	0.75	2.6068	2.7343	2.7996	2.8502	2.6843	2.7505	2.7658	2.7723
	1.00	3.3122	3.2546	3.2412	3.2146	3.2390	3.2250	3.2150	3.1879
	1.25	2.9583	2.8590	2.8013	2.7229	2.8153	2.7640	2.7369	2.6809
	1.50	1.7888	1.7637	1.7338	1.6801	1.6996	1.6931	1.6802	1.6359
1.00	0.50	1.7044	1.7768	1.7980	1.8110	1.7281	1.7393	1.7410	1.7417
	0.75	2.6313	2.6959	2.7250	2.7394	2.6832	2.7196	2.7240	2.7219
	1.00	3.3102	3.2491	3.2310	3.1991	3.2399	3.2221	3.2098	3.1798
	1.25	2.8829	2.8173	2.7738	2.7160	2.7831	2.7452	2.7281	2.6852
	1.50	1.7330	1.7215	1.7033	1.6683	1.6725	1.6697	1.6680	1.6372
1.40	0.50	1.6778	1.7259	1.7387	1.7457	1.7093	1.7170	1.7179	1.7182
	0.75	2.6518	2.6987	2.7178	2.7224	2.6886	2.7153	2.7162	2.7114
	1.00	3.3125	3.2528	3.2344	3.2024	3.2418	3.2238	3.2112	3.1807
	1.25	2.8593	2.8026	2.7652	2.7162	2.7704	2.7384	2.7255	2.6883
	1.50	1.7117	1.7054	1.6918	1.6644	1.6643	1.6632	1.6578	1.6380