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METHOD OF PRODUCT SPACINGS IN THE TWO–PARAMETER GAMMA DISTRIBUTION

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summary

The Gamma Distribution is widely used in Engineering and Industrial applications. Estimation of parameters is revisited in the two–parameter Gamma distribution. The method of product spacings is implemented to this distribution. A comparative study between the method of moments, the maximum likelihood method, and the method of product spacings is performed using simulation. For the scale parameter, the maximum likelihood estimate performs better and for the shape parameter, the product spacings estimate performs better.

Keywords and phrases: Di–gamma function; Newton–Raphson root finding method.

1 Introduction

The random variable X has a Gamma distribution with two parameters β and α if it has a probability density function of the form:

$$f(x;\beta,\alpha) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}; \quad \beta > 0, \quad \alpha > 0, \tag{1}$$

where α is known as the shape parameter and β as the scale parameter. The distribution function of the Gamma distribution (1) can be written as

$$F(x;\beta,\alpha) = \int_0^x \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^{\alpha}}dt; \quad \beta > 0, \quad \alpha > 0. \tag{2}$$

The random variables $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are defined as an ordered random sample from the Gamma distribution (1).

In the literature, estimation of parameters in the two parameter Gamma distribution is discussed extensively. Readers are referred to the following references: Harter and Moore (1965), Choi and Wette (1969), Wilks (1990), Lee (1992), Dang and Weerakkody (2000), and Evans et al., (2000). In this paper, the method of product spacings is implemented in estimating parameters in a two parameter Gamma distribution. The method of product spacings is compared with the method of moments and the method of maximum likelihood using simulation.

The organization of the paper is as follows: Different estimation procedures are presented in Section 2. In Section 3, a comparison study is conducted using simulation. An application is presented in Section 4. Finally, a concluding summary is presented in Section 5.

2 Estimation Procedures

2.1 Method of Moment Estimates (MME)

The method of moment estimates for β and α are respectively,

$$\hat{\beta}_M = \frac{S^2}{\bar{X}} \quad \text{and} \quad \hat{\alpha}_M = \left(\frac{\bar{X}}{S}\right)^2,$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

2.2 Maximum Likelihood Estimates (MLE)

The maximum likelihood estimates for β and α are respectively,

$$\hat{\beta}_L = \frac{\bar{X}}{\hat{\alpha}_L}$$

with $\hat{\alpha}_L$ found as the solution of the following non–linear equation

$$\log\hat{\alpha}_L - \Psi(\hat{\alpha}_L) = \log\left[\bar{X}/\left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}}\right] \tag{3}$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ and $\Gamma'(\alpha)$ is the derivative of $\Gamma(\alpha)$ with respect to α. $\Psi(\alpha)$ is also known as the Di–gamma function.

The solution of (3) can easily be obtained using the Newtom–Raphson method with $\hat{\alpha}_M$ as the starting value for $\hat{\alpha}_L$.

2.3 Method of Product Spacings (MPS)

The method of product spacings (MPS) was concurrently introduced by Cheng and Amin (1983) and Ranneby (1984). Let

$$D_i = \int_{x_{i-1:n}}^{x_{i:n}} f(x;\theta)dx, \quad i = 1, 2, \cdots, n+1,$$

where $x_{0:n}$ is the lower limit and $x_{n+1:n}$ is the upper limit of the domain of the density function $f(x;\theta)$, and θ can be vector–valued. Also, $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are defined as an ordered random sample from $f(x;\theta)$. Clearly, the spacings sum to unity, that is $\sum D_i = 1$. The MPS method is, quite simply, to choose θ to maximize the geometric mean of the spacings,

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

or, equivalently, its logarithm

$$H = \ln G.$$

MPS estimation gives consistent estimators under much more general conditions than MLEs. MPS estimators are asymptotically normal and are asymptotically as efficient as MLEs when these exist. For detailed goodness properties of MPS estimators, readers are referred to Cheng and Amin (1983), Ranneby (1984), Cheng and Iles (1987), Shah and Gokhale (1993), Rahman and Pearson (2002) and the references therein.

Using the density function (1) and the cdf (2), H can be written as follows:

$$\begin{aligned} H &= \tfrac{1}{n+1}\left[\ln F(X_{1:n};\beta,\alpha) + \ln\{1 - F(X_{n:n};\beta,\alpha)\}\right] \\ &\quad + \tfrac{1}{n+1}\left[\textstyle\sum_{i=1}^{n-1} \ln\{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)\}\right] \end{aligned} \tag{4}$$

By maximizing (4) for different values of β and α, the MPS estimates can be obtained as $\hat{\beta}_P$ and $\hat{\alpha}_P$. The Newton–Raphson method can be used in solving when the two first derivatives are equal to zero. The MME's are used as the starting values. The first derivatives of H with respect to β and α are respectively,

$$\begin{aligned} H'_\beta &= \tfrac{1}{n+1}\left[\tfrac{F'_\beta(X_{1:n};\beta,\alpha)}{F(X_{1:n};\beta,\alpha)} + \textstyle\sum_{i=1}^{n-1} \tfrac{F'_\beta(X_{i+1:n};\beta,\alpha) - F'_b(X_{i:n};\beta,\alpha)}{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)}\right. \\ &\quad \left. - \tfrac{F'_\beta(X_{n:n};\beta,\alpha)}{1 - F(X_{n:n};\beta,\alpha)}\right] \end{aligned} \tag{5}$$

and

$$\begin{aligned} H'_\alpha &= \tfrac{1}{n+1}\left[\tfrac{F'_\alpha(X_{1:n};\beta,\alpha)}{F(X_{1:n};\beta,\alpha)} + \textstyle\sum_{i=1}^{n-1} \tfrac{F'_\alpha(X_{i+1:n};\beta,\alpha) - F'_\alpha(X_{i:n};\beta,\alpha)}{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)}\right. \\ &\quad \left. - \tfrac{F'_\alpha(X_{n:n};\beta,\alpha)}{1 - F(X_{n:n};\beta,\alpha)}\right] \end{aligned} \tag{6}$$

where

$$F'_\beta(x;\beta,\alpha) = \frac{\alpha}{\beta}\left[F(x;\beta,\alpha+1) - F(x;\beta,\alpha)\right],$$

$$F'_\alpha(x;\beta,\alpha) = E_x(\ln x;\beta,\alpha) - F(x;\beta,\alpha)(\ln\beta + \Psi(\alpha)),$$

and

$$E_x(\ln x;\beta,\alpha) = \int_0^x \ln t \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}dt.$$

The second derivatives of H with respect to β and α are respectively,

$$\begin{aligned} H''_{\beta\beta} = & \tfrac{1}{n+1}\Big[\tfrac{F(X_{1:n};\beta,\alpha)F''_{\beta\beta}(X_{1:n};\beta,\alpha)-\left\{F'_\beta(X_{1:n};\beta,\alpha)\right\}^2}{\{F(X_{1:n};\beta,\alpha)\}^2} \\ & + \textstyle\sum_{i=1}^{n-1} \tfrac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\beta\beta}(X_{i+1:n};\beta,\alpha)-F''_{\beta\beta}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\left\{F'_\beta(X_{i+1:n};\beta,\alpha)-F'_\beta(X_{i:n};\beta,\alpha)\right\}^2}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\beta\beta}(X_{n:n};\beta,\alpha)+\left\{F'_\beta(X_{n:n};\beta,\alpha)\right\}^2}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Big], \end{aligned} \tag{7}$$

$$\begin{aligned} H''_{\beta\alpha} = & \tfrac{1}{n+1}\Big[\tfrac{F(X_{1:n};\beta,\alpha)F''_{\beta\alpha}(X_{1:n};\beta,\alpha)-F'_\beta(X_{1:n};\beta,\alpha)F'_\alpha(X_{1:n};\beta,\alpha)}{\{F(X_{1:n};\beta,\alpha)\}^2} \\ & + \textstyle\sum_{i=1}^{n-1} \tfrac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\beta\alpha}(X_{i+1:n};\beta,\alpha)-F''_{\beta\alpha}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\left\{F'_\beta(X_{i+1:n};\beta,\alpha)-F'_\beta(X_{i:n};\beta,\alpha)\right\}\left\{F'_\alpha(X_{i+1:n};\beta,\alpha)-F'_\alpha(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\beta\alpha}(X_{n:n};\beta,\alpha)+F'_\beta(X_{n:n};\beta,\alpha)F'_\alpha(X_{n:n};\beta,\alpha)}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Big] \end{aligned} \tag{8}$$

and

$$\begin{aligned} H''_{\alpha\alpha} = & \tfrac{1}{n+1}\Big[\tfrac{F(X_{1:n};\beta,\alpha)F''_{\alpha\alpha}(X_{1:n};\beta,\alpha)-\left\{F'_\alpha(X_{1:n};\beta,\alpha)\right\}^2}{\{F(X_{1:n};\beta,\alpha)\}^2} \\ & + \textstyle\sum_{i=1}^{n-1} \tfrac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\alpha\alpha}(X_{i+1:n};\beta,\alpha)-F''_{\alpha\alpha}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\left\{F'_\alpha(X_{i+1:n};\beta,\alpha)-F'_\alpha(X_{i:n};\beta,\alpha)\right\}^2}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\ & - \tfrac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\alpha\alpha}(X_{n:n};\beta,\alpha)+\left\{F'_\alpha(X_{n:n};\beta,\alpha)\right\}^2}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Big], \end{aligned} \tag{9}$$

where

$$F''_{\beta\beta}(x;\beta,\alpha) = \frac{\alpha(\alpha+1)}{\beta^2}\left[F(x;\beta,\alpha+2) - 2F(x;\beta,\alpha+1) + F(x;\beta,\alpha)\right],$$

$$\begin{aligned} F''_{\beta\alpha}(x;\beta,\alpha) = & \tfrac{\alpha}{\beta}\left[E_x(\ln x;\beta,\alpha+1) - E_x(\ln x;\beta,\alpha)\right] \\ & -\tfrac{\alpha}{\beta}F(x;\beta,\alpha+1)(\ln\beta + \Psi(\alpha) - \tfrac{1}{\beta}F(x;\beta,\alpha)(1-\alpha\ln\beta - \alpha\Psi(\alpha), \end{aligned}$$

$$\begin{aligned} F''_{\alpha\alpha}(x;\beta,\alpha) = & E_x((\ln x)^2;\beta,\alpha) - 2E_x(\ln x;\beta,\alpha)(\ln\beta + \Psi(\alpha)) \\ & +F(x;\beta,\alpha)\left[(\ln\beta)^2 + 2\ln\beta\Psi(\alpha) - \Psi'(\alpha) + \Psi(\alpha)\right], \end{aligned}$$

with $\Psi'(\alpha)$ being the derivative of $\Psi(\alpha)$, and

$$E_x((\ln x)^2;\beta,\alpha) = \int_0^x (\ln t)^2 \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}dt.$$

Then, the multivariate Newton–Raphson iteration is performed as

$$\begin{bmatrix} \hat{\beta}_P^{(l+1)} \\ \hat{\alpha}_P^{(l+1)} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_P^{(l)} \\ \hat{\alpha}_P^{(l)} \end{bmatrix} - \begin{bmatrix} H''^{(l)}_{\beta\beta} & H''^{(l)}_{\beta\alpha} \\ H''^{(l)}_{\beta\alpha} & H''^{(l)}_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} H'^{(l)}_{\beta} \\ H'^{(l)}_{\alpha} \end{bmatrix}, \tag{10}$$

where l is the index for the iterations.

3 Simulation Results

One thousand samples are generated for two different parameter settings $\{(\beta = 0.5, \alpha = 0.5)$ and $(\beta = 2.0, \alpha = 4.0)\}$ and for two different sample sizes ($n = 20$ and $n = 50$). Means (MEAN), standard deviations (SD), biases (BIAS), mean of the absolute biases (MAB) and mean squared errors (MSE) are computed and displayed in Table 1. MATLAB software is used in all computations and the codes are readily available.

4 Application

The following data in Table 2 represents failure times of machine parts from manufacturer A and are taken from $http : //v8doc.sas.com/sashtml/stat/chap29/sect44.htm$:

For this data, $\hat{\beta}_M = 483.22$, $\hat{\beta}_L = 550.60$, $\hat{\beta}_P = 604.13$, $\hat{\alpha}_M = 0.97$, $\hat{\alpha}_L = 0.85$, and $\hat{\alpha}_P = 0.80$.

5 Summary and Concluding Remarks

From Table 1, it is observed that all the estimates appear to be consistent and asymptotically unbiased. In terms of estimating β it should be noted that $\hat{\beta}_P$ has higher bias, mean absolute bias, standard deviation and mean squared error compared to $\hat{\beta}_M$ and $\hat{\beta}_L$. However, $\hat{\alpha}_P$ has lower bias, mean absolute bias, standard deviation and mean squared error compared to $\hat{\alpha}_M$ and $\hat{\alpha}_L$. Thus, the performance of $\hat{\beta}_L$ is better compared to $\hat{\beta}_M$ and $\hat{\beta}_P$ and $\hat{\alpha}_P$ is better compared to $\hat{\alpha}_M$ and $\hat{\alpha}_L$.

Such comparative studies are performed by implementing MPS method for the Weibull distribution by Rahman and Pearson (2003a), for the Pareto distribution by Rahman and Pearson (2003), for the Two–parameter Exponential distribution by Rahman and Pearson (2002), and for Burr XII distributions by Shah and Gokhale (1993).

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Table 1: Simulation Results

	$\hat{\beta}_M$	$\hat{\beta}_L$	$\hat{\beta}_P$	$\hat{\alpha}_M$	$\hat{\alpha}_L$	$\hat{\alpha}_P$
$\beta = 0.5$ $\alpha = 0.5$ $n = 20$						
MEAN	0.4383	0.4783	0.6780	0.6755	0.5657	0.4699
SD	0.2546	0.2046	0.4679	0.2725	0.1644	0.1337
BIAS	-0.0617	-0.0217	0.1780	0.1755	0.0657	-0.0301
MAB	0.1990	0.1602	0.2591	0.2334	0.1188	0.1024
MSE	0.0686	0.0423	0.2506	0.1050	0.0313	0.0188
$\beta = 0.5$ $\alpha = 0.5$ $n = 50$						
MEAN	0.4718	0.4782	0.5845	0.5756	0.5344	0.4714
SD	0.1755	0.1220	0.1865	0.1653	0.0846	0.0909
BIAS	-0.0282	-0.0218	0.0845	0.0756	0.0344	-0.0286
MAB	0.1409	0.0972	0.1400	0.1401	0.0669	0.0747
MSE	0.0316	0.0154	0.0419	0.0330	0.0083	0.0091
$\beta = 2.0$ $\alpha = 4.0$ $n = 20$						
MEAN	1.8816	1.9094	2.5214	4.7944	4.6453	3.6000
SD	0.6997	0.6368	0.8486	1.7524	1.6308	1.2461
BIAS	-0.1184	-0.0906	0.5214	0.7944	0.6453	-0.4000
MAB	0.5653	0.5195	0.7415	1.3696	1.2184	1.0451
MSE	0.5037	0.4137	0.9919	3.7021	3.0758	1.7128
$\beta = 2.0$ $\alpha = 4.0$ $n = 50$						
MEAN	1.9589	1.9749	2.2743	4.3179	4.2389	3.7237
SD	0.4639	0.4158	0.4814	0.9858	0.8708	0.7600
BIAS	-0.0411	-0.0251	0.2743	0.3179	0.2389	-0.2763
MAB	0.3705	0.3360	0.4292	0.7839	0.6873	0.6579
MSE	0.2168	0.1736	0.3070	1.0729	0.8153	0.6540

Table 2: Failure Times

620	470	260	89	388	242	103	100	39	460	284
1285	218	393	106	158	152	477	403	103	69	158
818	947	399	1274	32	12	134	660	548	381	203
871	193	531	317	85	1410	250	41	1101	32	421
32	343	376	1512	1792	47	95	76	515	72	1585
253	6	860	89	1055	537	101	385	176	11	565
164	16	1267	352	160	195	1279	356	751	500	803
560	151	24	689	1119	1733	2194	763	555	14	45
776	1									