

## EXPLICIT EXPRESSIONS FOR MOMENTS OF GENERALIZED GAMMA ORDER STATISTICS

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### SUMMARY

Explicit closed form expressions for moments of order statistics from the generalized gamma distribution are derived using Lauricella function of type A.

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## 1 Introduction

Suppose  $X_1, X_2, \dots, X_n$  is a random sample from the generalized gamma distribution given by the probability density function (pdf):

$$f(x) = \frac{cx^{c\alpha-1} \exp(-x^c)}{\Gamma(\alpha)} \quad (1.1)$$

for  $x > 0$ ,  $\alpha > 0$  and  $c > 0$ . Let  $X_{1:n} < X_{2:n} < \dots < X_{n:n}$  denote the corresponding order statistics. There has been some work relating to moments of the generalized gamma order statistics  $X_{r:n}$ , see e.g. Khan and Khan (1983) and Thomas (1996). However, all of the work that we are aware of express  $E(X_{r:n}^k)$  in terms of recurrence relations, numerical tables or multiple infinite series of the form

$$E(X_{r:n}^k) = \sum_{m_1=0}^{\infty} \dots \sum_{m_j=0}^{\infty} A(m_1, \dots, m_j, r, n, k) \quad (1.2)$$

for some function  $A(\cdot)$  and  $j = j(r, n, k)$ . These representations have little practical appeal because, for example, the computation of the  $j$ -fold infinite sum in (1.2) will become prohibitive as  $j$  gets large. Even for moderate  $j$ , one would have to code in (1.2) and this will be a waste of man power as well as computer time.

In this note, for the first time, we derive expressions for  $E(X_{r:n}^k)$  that are finite sums of a well known special function – namely, the Lauricella function of type A (Exton, 1978) defined by

$$F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{a_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n} x_1^{m_1} \dots x_n^{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n} m_1! \dots m_n!}, \quad (1.3)$$

where  $(f)_k = f(f+1)\dots(f+k-1)$  denotes the ascending factorial. Numerical routines for the direct computation of (1.3) are widely available, see e.g. Mathematica and Exton (1978).

## 2 Explicit Expression for $E(X_{r:n}^k)$

If  $X_1, X_2, \dots, X_n$  is a random sample from (1.1) then it is well known that the pdf of  $Y = X_{r:n}$  is given by

$$f_Y(y) = \frac{n!}{(r-1)!(n-r)!} \{F(y)\}^{r-1} \{1-F(y)\}^{n-r} f(y)$$

for  $r = 1, 2, \dots, n$ , where  $F(\cdot)$  is the cumulative distribution function (cdf) corresponding to (1.1) given by

$$F(y) = \frac{\gamma(\alpha, y^c)}{\Gamma(\alpha)},$$

where  $\gamma(\cdot, \cdot)$  denotes the incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \exp(-t) dt.$$

Thus, the  $k$ th moment of  $X_{r:n}$  can be expressed as

$$\begin{aligned}
 E(X_{r:n}^k) &= \frac{cn!}{(r-1)!(n-r)! \{\Gamma(\alpha)\}^n} \\
 &\quad \times \int_0^\infty y^{k+c\alpha-1} \exp(-y^c) \{\gamma(\alpha, y^c)\}^{r-1} \{\Gamma(\alpha) - \gamma(\alpha, y^c)\}^{n-r} dy \\
 &= \frac{cn!}{(r-1)!(n-r)! \{\Gamma(\alpha)\}^n} \int_0^\infty y^{k+c\alpha-1} \exp(-y^c) \\
 &\quad \times \sum_{l=0}^{n-r} \binom{n-r}{l} \{\Gamma(\alpha)\}^{n-r-l} (-1)^l \{\gamma(\alpha, y^c)\}^{r+l-1} dy \\
 &= \frac{cn!}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} \\
 &\quad \times \int_0^\infty y^{k+c\alpha-1} \exp(-y^c) \{\gamma(\alpha, y^c)\}^{r+l-1} dy \\
 &= \frac{cn!}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} I(l). \tag{2.1}
 \end{aligned}$$

Using the series expansion

$$\gamma(\alpha, x) = x^\alpha \sum_{m=0}^\infty \frac{(-x)^m}{(\alpha+m)m!},$$

the integral  $I(l)$  in (2.1) can be expressed as

$$\begin{aligned}
 I(l) &= \int_0^\infty y^{k+c\alpha-1} \exp(-y^c) \left\{ y^{c\alpha} \sum_{m=0}^\infty \frac{(-y^c)^m}{(\alpha+m)m!} \right\}^{r+l-1} dy \\
 &= \int_0^\infty \sum_{m_1=0}^\infty \cdots \sum_{m_{r+l-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+l-1}} y^{k+c\alpha(r+l)+c(m_1+\cdots+m_{r+l-1})-1} \exp(-y^c)}{(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!} dy \\
 &= \sum_{m_1=0}^\infty \cdots \sum_{m_{r+l-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+l-1}}}{(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!} \\
 &\quad \times \int_0^\infty y^{k+c\alpha(r+l)+c(m_1+\cdots+m_{r+l-1})-1} \exp(-y^c) dy \\
 &= \sum_{m_1=0}^\infty \cdots \sum_{m_{r+l-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+l-1}} \Gamma(k/c + \alpha(r+l) + m_1 + \cdots + m_{r+l-1})}{c(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!}. \tag{2.2}
 \end{aligned}$$

Using the fact  $(f)_k = \Gamma(f+k)/\Gamma(f)$  and the definition in (1.3), one can reexpress (2.2) as

$$\begin{aligned}
 I(l) &= c^{-1} \alpha^{1-r-l} \Gamma(k/c + \alpha(r+l)) \\
 &\quad \times F_A^{(r+l-1)}(k/c + \alpha(r+l), \alpha, \dots, \alpha; \alpha+1, \dots, \alpha+1; -1, \dots, -1). \tag{2.3}
 \end{aligned}$$

Combining (2.1) and (2.3), we obtain the expression

$$E(X_{r:n}^k) = \frac{n!}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} \alpha^{1-r-l} \Gamma(k/c + \alpha(r+l)) \\ \times F_A^{(r+l-1)}(k/c + \alpha(r+l), \alpha, \dots, \alpha; \alpha+1, \dots, \alpha+1; -1, \dots, -1) \quad (2.4)$$

Note that (2.4) is a finite sum of the Lauricella function of type A.

### 3 Conclusions

We have derived an expression for moments of the generalized gamma order statistics as a finite sum of a well known special function. This expression is simpler than previously known work.

### References

- [1] Exton, H. (1978). *Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs*. Halsted Press, New York.
- [2] Khan, A. H. and Khan, R. U. (1983). Recurrence relations between moments of order statistics from generalized gamma distribution. *Journal of Statistical Research*, **17**, 75–82.
- [3] Thomas, P. Y. (1996). On the moments of extremes from generalized gamma distribution. *Communications in Statistics—Theory and Methods*, **25**, 1825–1836.