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EXACT VALUES OF EXPECTED VALUES, VARIANCES AND COVARIANCES OF ORDER STATISTICS FROM THE STUDENT'S t DISTRIBUTION

A. Adatia
Department of Mathematics and Computer Science
University of Lethbridge, Lethbridge, Canada
Email: aminmohamed.adatia@uleth.ca

SUMMARY

Expressions of exact moments and product moments of order statistics from the Student's t distribution with υ degrees of freedom are obtained. Tables of expected values and variances for sample size n = 20(5)30 and covariances for n = 20 when $\upsilon = 3$ are given.

Keywords and phrases: t distribution, order statistics, moments

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1 Introduction

Estimation of location and scale parameters based on order statistics was introduced by Lloyd (1952). It has the advantage over more traditional estimation procedures that the estimators are always unbiased and of minimum variance in the class of unbiased linear estimators. More over the more traditional estimation procedures are often very tedious to obtain and the estimators are not usually in a closed form. Thus the small sample properties of the estimators are difficult to obtain. However, to construct the estimators based on order statistics and to investigate their properties, the values of the expected values and covariances of the order statistics are required. Student's t distribution with υ degrees of freedom

$$f(x,\upsilon) = \frac{\left(1+\frac{x^2}{\upsilon}\right)^{-(\frac{\upsilon+1}{2})}}{\sqrt{\upsilon}B[1/2,\upsilon/2]}$$

has been extensively studied and a comprehensive treatment of the distribution is available in Johnson et al. (1995). Tiku and Kumra (1985) have obtained approximate values of expected values and variances and covariances of order statistics for Student's t distribution. The values are given for p = $(\upsilon+ 1)/2 = 2(0.5)10$ and sample size ≤ 20. These values are obtained by employing numerical integration techniques and are thus only approximate. Also, the values for sample size larger than 20 are not available. In this paper, expressions of exact moments and product moments of order statistics from the Student's t distribution with υ degrees of freedom are obtained and are then utilized to construct tables of expected values and variances for sample size n=20(5)30 and covariances for n=20 when $\upsilon = 3$. More extensive tables are available from the author upon request.

2 Moments of Order Statistics

The r^{th} moment for the i^{th} order statistic from a random sample of size n from a Student's t distribution is given by

$$E[X_{i:n}^r] = \frac{n!}{(i-1)!(n-i)!}\int_{-\infty}^{\infty} x^r F(x,\upsilon)^{i-1}\{1-F(x,\upsilon)\}^{n-i} f(x,\upsilon)dx$$

Let $\theta_x = \arctan(x/\sqrt{\upsilon})$, then $F(x,\upsilon)$ is given by Zelen and Severo (1964).

$$\begin{aligned} F(\theta_x,\upsilon) &= \frac{1}{2}\left(1+\sum_{m=0}^{\frac{\upsilon}{2}-1} \mathrm{Si}\,n(\theta_x)\binom{2m}{m}\left(\frac{Cos^2(\theta_x)}{4}\right)^m\right), \upsilon \geq 2 \text{ and even} \\ F(\theta_x,\upsilon) &= \frac{1}{2}+\frac{1}{\pi}\left(\theta_x + \mathrm{Si}\,n(\theta_x)\sum_{m=0}^{\frac{(\upsilon-3)}{2}} \frac{4^m}{(2m+1)\binom{2m}{m}} Cos^{2m+1}(\theta_x)\right), \upsilon \geq 3 \text{ and odd.} \end{aligned}$$

The r^{th} moment for the i^{th} order statistic when $\upsilon \geq 2$ and even is given by

$$\begin{aligned} E[X_{i:n}^r] &= \frac{n!}{(i-1)!(n-i)!}\left(\frac{\upsilon^{r/2}}{B[1/2,\upsilon/2]}\right)\sum_{k_1}^{n-i}\binom{n-j}{k_1}(-1)^{k_1}(\frac{1}{2})^{i-1+k_1} \\ &\times \sum_{lx_{00}+\sum_{m=0}^{\upsilon/2-1} lx_m = i-1+k1} \begin{pmatrix} & & i-1+k_1 & \\ lx_{00} & lx_0 & \cdots & lx_{\upsilon/2-1} \end{pmatrix} \\ &\times \left(\prod_{m=0}^{\upsilon/2-1}\left\{\binom{2m}{m}\left(\frac{1}{4}\right)^m\right\}^{lx_m}\right)\left(1+(-1)^{r+\sum_{m=0}^{\upsilon/2-1} lx_m}\right) \\ &\times B\left[(1+r+\sum_{m=0}^{\upsilon/2-1} lx_m)/2, (\upsilon - r + 2\sum_{m=0}^{\upsilon/2-1} m lx_m)/2\right] \end{aligned}$$

The r^{th} moment for the i^{th}order statistic when $\upsilon \geq 3$ and odd is given by

$$
\begin{aligned}
E[X_{i:n}^r] &= \frac{n!}{(i-1)!(n-i)!} \times \frac{\upsilon^{r/2}}{B[1/2,\upsilon/2]} \sum_{k_1}^{n-i} \binom{n-j}{k_1} (-1)^{k_1} \left(\frac{1}{2}\right)^{i-1+k_1} \\
& \sum_{lx_{000}+lx_{00}+\sum_{m=0}^{(\upsilon-3)/2} lx_m = i-1+k_1} \begin{pmatrix} & & i-1+k_1 & & \\ lx_{000} & lx_{00} & lx_0 & \ldots & lx_{(\upsilon-3)/2} \end{pmatrix} \\
& \left(\prod_{m=0}^{(\upsilon-3)/2} \left(\frac{4^m}{(2m+1)\binom{2m}{m}} \right)^{lx_m} \right) \left(\frac{2}{\pi}\right)^{lx_{00}+\sum_{m=0}^{(\upsilon-3)/2} lx_m} \\
& g(lx_{00}, r + \sum_{m=0}^{(\upsilon-3)/2} lx_m, (\upsilon-1-r) + \sum_{m=0}^{(\upsilon-3)/2} (2m+1) lx_m),
\end{aligned}
$$

where

$$
\begin{aligned}
g(p,ns,mc) &= \left(\frac{1}{2}\right)^{mc} \sum_{s1=0}^{ns} \binom{ns}{s1} \sum_{c1=0}^{mc} \binom{mc}{c1} (-1)^{s1} g1(p, \sqrt{-1}(ns+mc-2s1-2c1))/(2\sqrt{-1})^{ns} \\
g1(p,q) &= \begin{cases} 2^{-(1+p)}\left(1+(-1)^p\right)\pi^{1+p}/(1+p) & \text{when } q=0 \text{ and } p>-1 \\ \sum_{j=0}^{p} p! \exp(pq) \left\{(\pi/2)^{p-j} - (-\pi/2)^{p-j}\right\} (-1)^j q^{-1-j}/\{(p-j)!\}, & \text{when } q \neq 0 \end{cases}
\end{aligned}
$$

Product moment of i^{th} and j^{th} order statistics is given by

$$
\begin{aligned}
E[X_{i:n}X_{j:n}] &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \iint_{-\infty<x<y<\infty} xyF(x,\upsilon)^{i-1}\{F(y,\upsilon)-F(x,\upsilon)\}^{j-i-1} \\
& \times\{1-F(y,\upsilon)\}^{n-j} f(x,\upsilon) f(y,\upsilon) dydx, \ 1 \le i \le j \le n.
\end{aligned}
$$

Thus the product moment of i^{th} and j^{th} order statistics when $\upsilon \geq 2$ and even is given by

$$
\begin{aligned}
& E[X_{i:n}X_{j:n}] = \frac{(n!)\,\upsilon\,\{B[1/2,\upsilon/2]\}^{-2}}{(i-1)!(j-i-1)!(n-j)!} \sum_{k_0}^{j-i-1} \sum_{k_1}^{n-j} \binom{j-i-1}{k_0}\binom{n-j}{k1} (-1)^{j-i-1-k_0+k_1} \left(\tfrac{1}{2}\right)^{k_0+k_1} \\
& \times \sum_{ly_{00}+\sum_{m=0}^{(\upsilon-2)/2} ly_m = k_0+k_1} \begin{pmatrix} & & k_0+k_1 & \\ ly_{00} & ly_0 & \ldots & ly_{\upsilon/2-1} \end{pmatrix} \left(\prod_{m=0}^{\upsilon/2-1} \left(\binom{2m}{m}\left(\tfrac{1}{4}\right)^m\right)^{ly_m} \right) \left(\tfrac{1}{2}\right)^{j-2-k_0} \\
& \sum_{lx_{00}+\sum_{m=0}^{(\upsilon-2)/2} lx_m = j-2-k_0} \begin{pmatrix} & & j-2-k_0 & \\ lx_{00} & lx_0 & \ldots & lx_{\upsilon/2-1} \end{pmatrix} \left(\prod_{m=0}^{\upsilon/2-1} \left(\binom{2m}{m}\left(\tfrac{1}{4}\right)^m\right)^{lx_m} \right) \\
& \times gp\left(0,0,1+\sum_{m=0}^{\upsilon/2-1} lx_m, (\upsilon-2)+2\sum_{m=0}^{\upsilon/2-1} m\,lx_m, 1+\sum_{m=0}^{\upsilon/2-1} ly_m, (\upsilon-2)+2\sum_{m=0}^{\upsilon/2-1} m\,ly_m\right)
\end{aligned}
$$

and the product moment of i^{th} and j^{th} order statistics when $\upsilon \geq 3$ and odd is given by

$$E[X_{i:n}X_{j:n}] = \frac{(n!)\,\upsilon\,\{B[1/2,\upsilon/2]\}^{-2}}{(i-1)!(j-i-1)!(n-j)!}\sum_{k_0}^{j-i-1}\sum_{k_1}^{n-j}\binom{j-i-1}{k_0}\binom{n-j}{k1}(-1)^{j-i-1-k_0+k_1}\left(\tfrac{1}{2}\right)^{k_0+k_1}$$
$$\times \sum_{ly_{000}+ly_{00}+\sum_{m=0}^{(\upsilon-3)/2} ly_m = k_0+k_1} \begin{pmatrix} & & & k_0+k_1 & \\ ly_{000} & ly_{00} & ly_0 & \cdots & ly_{(\upsilon-3)/2}\end{pmatrix}\left(\tfrac{1}{2}\right)^{j-2-k_0}$$
$$\times \sum_{lx_{000}+lx_{00}+\sum_{m=0}^{(\upsilon-3)/2} lx_m = j-2-k_0} \begin{pmatrix} & & & j-2-k_0 & \\ lx_{000} & lx_{00} & lx_0 & \cdots & lx_{(\upsilon-3)/2}\end{pmatrix}$$
$$\times\left(\left(\tfrac{2}{\pi}\right)^{ly_{00}+lx_{00}}\prod_{m=0}^{(\upsilon-3)/2}\left(\frac{2}{\pi}\frac{4^m}{(2m+1)\binom{2m}{m}}\right)^{ly_m+lx_m}\right)$$
$$\times gp\Bigg(lx_{00}+\sum_{m=0}^{(\upsilon-3)/2} lx_m, ly_{00}+\sum_{m=0}^{(\upsilon-3)/2} ly_m, 1+\sum_{m=0}^{(\upsilon-3)/2} lx_m, (\upsilon-2)$$
$$+\sum_{m=0}^{(\upsilon-3)/2}(2m+1)lx_m, 1+\sum_{m=0}^{(\upsilon-3)/2} ly_m, (\upsilon-2)+\sum_{m=0}^{(\upsilon-3)/2}(2m+1)ly_m\Bigg)$$

where

$$gp(p_{xx}, p_{yy}, p_x, q_x, p_y, q_y) = \sum_{p_{x0}=0}^{p_x}\sum_{q_{x0}=0}^{q_x}\sum_{p_{y0}=0}^{p_y}\sum_{q_{y0}=0}^{q_y}$$
$$\times\left(\frac{1}{2\sqrt{-1}}\right)^{p_x+p_y}\left(\frac{1}{2}\right)^{q_x+q_y}\binom{p_x}{p_{x0}}\binom{q_x}{q_{x0}}\binom{p_y}{p_{y0}}\binom{q_y}{q_{y0}}(-1)^{p_x+p_y-p_{x0}-p_{y0}}$$
$$\times\, gp1(p_{xx}, p_{yy}, \sqrt{-1}(2p_{x0}+2q_{x0}-p_x-q_x), \sqrt{-1}(2p_{y0}+2q_{y0}-p_y-q_y))$$

and

$$gp1(p_{xx},p_{yy},k,m)=$$
$$\begin{cases}\pi/2\ , \quad \text{if } p_{xx}=p_{yy}=k=m=0\\ (e^{m\pi/2}m\pi-2\,\mathrm{Si}\,nh(m\pi/2))/m^2, \quad \text{if } p_{xx}=p_{yy}=k=0 \text{ and } m\neq 0\\ e^{-k\pi/2}\left(-1+e^{k\pi}-k\pi\right)/k^2, \quad \text{if } p_{xx}=p_{yy}=m=0 \text{ and } k\neq 0\\ (2/\pi)^{-2-p_{yy}}\left(3+(-1)^{p_{yy}}+2p_{yy}\right)/\left(2+3p_{yy}+p_{yy}^2\right), \quad \text{if } \ p_{xx}=k=m=0 \text{ and } p_{yy}\neq 0\\ f_{12}(p_{yy},m) \quad \text{if } \ p_{xx}=k=0 \text{ and } p_{yy}\neq 0,\ m\neq 0\\ \left(\frac{\pi}{2}\right)^{p_{yy}+1}\frac{(e^{k\pi/2}-e^{-k\pi/2})}{k(p_{yy}+1)}-\frac{f_{11}(p_{yy}+1,k)}{(p_{yy}+1)} \quad \text{if } \ p_{xx}=m=0 \text{ and } p_{yy}\neq 0, \mathrm{k}\neq 0\\ \frac{\left((\pi/2)^{p_{xx}+1}-(-\pi/2)^{p_{xx}+1}\right)e^{m\pi/2}}{m(p_{xx}+1)}-\frac{f_{11}(p_{xx},m)}{m} \quad \text{if } \ p_{yy}=k=0 \text{ and } \ p_{xx}\neq 0, \mathrm{m}\neq 0\\ \frac{f_{11}(p_{xx},k)e^{m\pi/2}}{m}-\frac{\left((\pi/2)^{p_{xx}+1}-(-\pi/2)^{p_{xx}+1}\right)}{m(p_{xx}+1)} \quad \text{if } \ p_{xx}\neq 0, k\neq 0, m\neq 0 \text{ and } p_{yy}=m+k=0\\ f_{13}(p_{yy},m,k) \quad \text{if } p_{xx}=0, \ \ p_{yy}\neq 0, k\neq 0, m\neq 0 \text{ and } m+k\neq 0\\ \frac{e^{m\pi/2}(e^{k\pi/2}-e^{-k\pi/2})}{mk}-\frac{\pi}{m} \quad \text{if } k\neq 0, m\neq 0 \text{ and } p_{xx}=p_{yy}=m+k=0\\ \frac{e^{m\pi/2}(e^{\frac{k\pi}{2}}-e^{\frac{-k\pi}{2}})}{mk}-\frac{(e^{\frac{(k+m)\pi}{2}}-e^{\frac{-(k+m)\pi}{2}})}{m(k+m)} \quad \text{if } p_{xx}=p_{yy}=0, k\neq 0, m\neq 0 \text{ and } m+k\neq 0\\ \frac{4^{-(1+p_{xx})}\pi^{2+p_{xx}}(3(-2)^{p_{xx}}+2^{p_{xx}}(1+2(-1)^{p_{xx}}p_{xx}))}{(1+p_{xx})(2+p_{xx})} \quad \text{if } p_{xx}\neq 0, \ \ p_{yy}=0, k=0, m=0\\ (\pi(f_{11}(p_{xx},k)/2)-f_{11}(p_{xx}+1,k) \quad \text{if } p_{xx}\neq 0, \ \ p_{yy}=0, k\neq 0, m=0\end{cases}$$

where

$$\begin{aligned} f_1(m,a,x) &= \frac{e^{ax}}{a}\left(\sum_{n=0}^{m}\frac{m!}{(m-n)!}\frac{x^{m-n}}{a^n}(-1)^n\right)\\ f_{11}(x,m) &= f_1(x,m,\pi/2)-f_1(x,m,-\pi/2)\\ f_{12}(p_{yy},m) &= (\pi^2/2)\,f_1(p_{yy},m,\pi/2)-F_1(p_{yy},m,0)\\ f_{13}(p_{yy},m,k) &= (1/k)f_1(p_{yy},m,\pi/2)(e^{k\pi/2}-e^{-k\pi/2})-F_1(p_{yy},m,k)\\ F_1(p_{yy},m,k) &= \sum_{p_{yyy}=0}^{p_{yy}}\frac{p_{yy}!}{(p_{yy}-p_{yyy})!}(-1)^{p_{yyy}}\left(\frac{1}{m}\right)^{(p_{yyy}+1)} f_{11}(p_{yy}-p_{yyy},m+k)\end{aligned}$$

Example 2.1. Exact values of expected values, product moment and covariance of first and second order statistics from the Student's t distribution with degrees of freedom 3 and a sample of size 5, i.e., $n=5$, $\upsilon=3$, $i=1$ and $j=2$ are

$$E[X_{1:5}^1]=\tfrac{5\sqrt{3}(-35+24\pi^2)}{32\pi^3}, \quad E[X_{2:5}^1]=-\tfrac{175\sqrt{3}}{16\pi^3}, \quad E[X_{1:5}X_{2:5}]=\tfrac{5(-539+480\pi^2)}{128}$$
$$\text{and} \quad Cov[X_{1:5},X_{2:5}]=E[X_{1:5}X_{2:5}]-E[X_{1:5}^1]E[X_{2:5}^1]=\tfrac{91875-73780\pi^2+9600\pi^4}{512\pi^6}$$

3 Tables

Expected values and variances of the order statistics for $n = 20(5)30$ and $\upsilon = 3$ are tabulated in Table 1 and the values of the covariances are tabulated in Table 2 for $n = 20$ and $\upsilon = 3$. The variance for the i^{th} order statistic is

$$Var[X_{i:n}] = E[X_{i:n}^2] - E[X_{i:n}^1]^2, 1 \leq i \leq n.$$

The missing values in Table I and Table II can be obtained by using the following equations:

$$E[X_{i:n}^1] = E[X_{n-i+1:n}^1], \quad Var[X_{i:n}^1] = Var[X_{n-i+1:n}^1]$$
$$\text{and } Cov[X_{i:n}, X_{j:n}] = Cov[X_{n-i+1:n}, X_{n-j+1:n}] = Cov[X_{n-j+1:n}, X_{n-i+1:n}]$$

There are many applications of these tables (Balakrishnan and Cohen (1991)). One important application is the Lloyd's (1952) best linear unbiased estimators (BLUEs) of location and scale parameters.

Example 3.1. Let $\underline{Y}$ represent a vector of an ordered sample of size n from a t distribution with υ degrees of freedom and a location parameter μ and a scale parameter μ. The BLUEs of μ and μ are

$$\widehat{\mu} = \frac{\underline{1}'\underline{V}^{-1}\underline{Y}}{\underline{1}'\underline{V}^{-1}\underline{1}} \quad \text{and} \quad \widehat{\sigma} = \frac{\underline{\alpha}'\underline{V}^{-1}\underline{Y}}{\underline{\alpha}'\underline{V}^{-1}\underline{\alpha}}$$

where $\underline{1}' = (1, \cdots, 1), \underline{\alpha}' = (E[X_{1:n}^1], \cdots, E[X_{n:n}^1])$ and $\underline{V}^{-1}$ is an inverse of variance-covariance matrix. These values are obtained from Table 1 and Table 2.

In order to illustrate the method, an ordered random sample of size $n = 20$ was generated from a t distribution with $\upsilon = 3$, $\mu = 50$ and $\sigma = 10$.

$$\begin{aligned} \underline{Y'} \;=\; & (16.48, 36.82, 40.05, 40.47, 40.57, 42.02, 44.35, 45.20, 45.33, \\ & 45.35, 46.24, 52.38, 53.09, 55.43, 57.74, 58.96, 60.70, 64.05, 65.32, 78.55) \end{aligned}$$

Suppose we assumed that μ and σ are unknown and are to be estimated. Then the BLUEs are $\widehat{\mu} = 48.74$ and $\widehat{\sigma} = 9.264$.

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Table 1: Expected values and variances of order statistics when $\upsilon = 3$

n	i	$E[X^1_{i:n}]$	$Var[X_{i:n}]$	n	i	$E[X^1_{i:n}]$	$Var[X_{i:n}]$	n	i	$E[X^1_{i:n}]$	$Var[X_{i:n}]$
20	1	-3.41748	7.17404	25	4	-1.33863	0.24022	30	4	-1.51468	0.25486
20	2	-1.99319	0.81165	25	5	-1.09403	0.17227	30	5	-1.26773	0.17976
20	3	-1.45694	0.36127	25	6	-0.89939	0.13539	30	6	-1.07378	0.13888
20	4	-1.12524	0.22678	25	7	-0.73511	0.11306	30	7	-0.91236	0.1139
20	5	-0.88008	0.16688	25	8	-0.59065	0.09863	30	8	-0.77254	0.09747
20	6	-0.68082	0.13481	25	9	-0.45961	0.08901	30	9	-0.64776	0.08614
20	7	-0.50871	0.11601	25	10	-0.33775	0.08259	30	10	-0.5338	0.0781
20	8	-0.3534	0.10466	25	11	-0.22204	0.07847	30	11	-0.42773	0.07233
20	9	-0.20834	0.0981	25	12	-0.11011	0.07616	30	12	-0.3274	0.06822
20	10	-0.06886	0.09508	25	13	0	0.07542	30	13	-0.23116	0.06539
25	1	-3.73892	8.23373	30	1	-4.01655	9.22626	30	14	-0.13765	0.06362
25	2	-2.23228	0.90458	30	2	-2.43554	0.99303	30	15	-0.04571	0.06276
25	3	-1.67615	0.39239	30	3	-1.85977	0.42326				

Table 2: Covariances of order statistics when $\upsilon = 3$

n	i	j	$Cov[X_{i:n}, X_{j:n}]$	n	i	j	$Cov[X_{i:n}, X_{j:n}]$	n	i	j	$Cov[X_{i:n}, X_{j:n}]$
20	1	2	1.11491	20	2	16	0.052132	20	4	15	0.048981
20	1	3	0.538715	20	2	17	0.051432	20	4	16	0.047403
20	1	4	0.344296	20	2	18	0.052301	20	4	17	0.046789
20	1	5	0.249991	20	2	19	0.05648	20	4	18	0.047602
20	1	6	0.195311	20	2	20	0.075165	20	4	19	0.051429
20	1	7	0.160052	20	3	4	0.233506	20	4	20	0.068473
20	1	8	0.135671	20	3	5	0.170669	20	5	6	0.131497
20	1	9	0.117973	20	3	6	0.133919	20	5	7	0.108418
20	1	10	0.104677	20	3	7	0.110079	20	5	8	0.092323
20	1	11	0.094449	20	3	8	0.093523	20	5	9	0.080565
20	1	12	0.086469	20	3	9	0.081467	20	5	10	0.071689
20	1	13	0.080219	20	3	10	0.072387	20	5	11	0.064835
20	1	14	0.075383	20	3	11	0.065389	20	5	12	0.059473
20	1	15	0.07179	20	3	12	0.059922	20	5	13	0.055266
20	1	16	0.069422	20	3	13	0.055636	20	5	14	0.052009
20	1	17	0.068473	20	3	14	0.052318	20	5	15	0.049593
20	1	18	0.069544	20	3	15	0.049856	20	5	16	0.048009
20	1	19	0.075159	20	3	16	0.048238	20	5	17	0.047403
20	1	20	0.099999	20	3	17	0.047602	20	5	18	0.048238
20	2	3	0.397131	20	3	18	0.048417	20	5	19	0.052132
20	2	4	0.255318	20	3	19	0.052298	20	5	20	0.069422
20	2	5	0.186025	20	3	20	0.069614	20	6	7	0.111336
20	2	6	0.145664	20	4	5	0.16627	20	6	8	0.094928
20	2	7	0.119557	20	4	6	0.130737	20	6	9	0.08292
20	2	8	0.101464	20	4	7	0.107624	20	6	10	0.073844
20	2	9	0.088309	20	4	8	0.09154	20	6	11	0.066828
20	2	10	0.078413	20	4	9	0.079809	20	6	12	0.061335
20	2	11	0.070793	20	4	10	0.070963	20	6	13	0.057024
20	2	12	0.064843	20	4	11	0.06414	20	6	14	0.053685
20	2	13	0.060182	20	4	12	0.058805	20	6	15	0.05121
20	2	14	0.056574	20	4	13	0.054621	20	6	16	0.049593
20	2	15	0.053895	20	4	14	0.051383	20	6	17	0.04898

Table 2: Covariances of order statistics when $\upsilon = 3$

n	i	j	$Cov[X_{i:n}, X_{j:n}]$	n	i	j	$Cov[X_{i:n}, X_{j:n}]$	n	i	j	$Cov[X_{i:n}, X_{j:n}]$
20	6	18	0.049856	20	9	14	0.064209	20	12	19	0.088309
20	6	19	0.053895	20	9	15	0.061335	20	12	20	0.117973
20	6	20	0.07179	20	9	16	0.059473	20	13	14	0.099048
20	7	8	0.099048	20	9	17	0.058805	20	13	15	0.094928
20	7	9	0.086613	20	9	18	0.059922	20	13	16	0.092323
20	7	10	0.0772	20	9	19	0.064843	20	13	17	0.09154
20	7	11	0.069917	20	9	20	0.086469	20	13	18	0.093523
20	7	12	0.064209	20	10	11	0.086351	20	13	19	0.101464
20	7	13	0.059728	20	10	12	0.079491	20	13	20	0.135671
20	7	14	0.056257	20	10	13	0.074096	20	14	15	0.111336
20	7	15	0.053685	20	10	14	0.069917	20	14	16	0.108418
20	7	16	0.052009	20	10	15	0.066828	20	14	17	0.107624
20	7	17	0.051382	20	10	16	0.064835	20	14	18	0.110079
20	7	18	0.052318	20	10	17	0.06414	20	14	19	0.119557
20	7	19	0.056574	20	10	18	0.065389	20	14	20	0.160052
20	7	20	0.075383	20	10	19	0.070793	20	15	16	0.131497
20	8	9	0.091626	20	10	20	0.094449	20	15	17	0.130737
20	8	10	0.081748	20	11	12	0.08762	20	15	18	0.133918
20	8	11	0.074096	20	11	13	0.081748	20	15	19	0.145664
20	8	12	0.068095	20	11	14	0.0772	20	15	20	0.195311
20	8	13	0.06338	20	11	15	0.073844	20	16	17	0.16627
20	8	14	0.059728	20	11	16	0.071689	20	16	18	0.17067
20	8	15	0.057024	20	11	17	0.070963	20	16	19	0.186025
20	8	16	0.055266	20	11	18	0.072387	20	16	20	0.249991
20	8	17	0.054621	20	11	19	0.078413	20	17	18	0.233508
20	8	18	0.055636	20	11	20	0.104677	20	17	19	0.255318
20	8	19	0.060182	20	12	13	0.091626	20	17	20	0.344296
20	8	20	0.080219	20	12	14	0.086613	20	18	19	0.397131
20	9	10	0.08762	20	12	15	0.08292	20	18	20	0.538717
20	9	11	0.079491	20	12	16	0.080565	20	19	20	1.11491
20	9	12	0.073111	20	12	17	0.079809				
20	9	13	0.068095	20	12	18	0.081467				