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# QUANTILE ESTIMATION IN THE TWO-PARAMETER GAMMA DISTRIBUTION

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SUMMARY

The gamma distribution is applicable in situations where intervals between events are considered as well as where a skewed distribution is appropriate. Estimation of parameters is revisited in the two-parameter gamma distribution. The method of quantile estimates is implemented to this distribution. A comparative study between the method of moments, the maximum likelihood method, the method of product spacings, and the method of quantile estimates is performed using simulation. For the scale parameter, the maximum likelihood estimate performs better and for the shape parameter, the product spacings estimate performs better.

*Keywords and phrases:* Di-gamma function; Newton-Raphson root finding method.

*AMS Classification:* 62F10

## 1 Introduction

The random variable $X$ has a gamma distribution with two parameters $\beta$ and $\alpha$ if it has a probability density function of the form:

$$f(x;\beta,\alpha) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}; \quad \beta > 0, \quad \alpha > 0, \tag{1.1}$$

where $\alpha$ is known as the shape parameter and $\beta$ as the scale parameter. The distribution function of the gamma distribution (1.1) can be written as

$$F(x;\beta,\alpha) = \int_0^x \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^{\alpha}}dt; \quad \beta > 0, \quad \alpha > 0. \tag{1.2}$$

The random variables $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are defined as an ordered random sample from the gamma distribution (1.1).

The gamma distribution is widely utilized in practice. For example, it is used in engineering and industry (life-time, flow analysis, and serving time distributions), climatology (rainfall amounts), and risk management(insurance claims and loan defaults).

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Estimation of parameters in the two-parameter gamma distribution has been investigated by numerous individuals. The following references provide a basic review of past results: Harter and Moore (1965), Choi and Wette (1969), Wilks (1990), Lee (1992), Dang and Weerakkody (2000), Evans et al. (2000), and Rahman et al. (2007). In this paper, the method of quantile estimates is implemented in estimating parameters in a two-parameter gamma distribution. The quantile estimates of the parameters are compared with parameter estimates based on the method of moments, the method of maximum likelihood, and the method of product spacings using simulation.

The organization of the paper is as follows: Different estimation procedures are presented in Section 2, In Section 3, a comparison study is conducted using simulation. An application is presented in Section 4, and a concluding summary is presented in Section 5.

# 2 Estimation Procedures

## 2.1 Method of Moment Estimates (MME)

The method of moment estimates for $\beta$ and $\alpha$ are respectively,

$$\hat{\beta}_M = \frac{S^2}{\bar{X}} \quad \text{and} \quad \hat{\alpha}_M = \left(\frac{\bar{X}}{S}\right)^2,$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

## 2.2 Maximum Likelihood Estimates (MLE)

The maximum likelihood estimates for $\beta$ and $c$ are respectively,

$$\hat{\beta}_L = \frac{\bar{X}}{\hat{\alpha}_L}$$

with $\hat{\alpha}_L$ found as the solution of the following non-linear equation

$$\log \hat{\alpha}_L - \Psi(\hat{\alpha}_L) = \log \left[ \bar{X} / \left( \prod_{i=1}^{n} X_i \right)^{\frac{1}{n}} \right], \tag{2.1}$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ and $\Gamma'(\alpha)$ is the derivative of $\Gamma(\alpha)$ with respect to $\alpha$. $\Psi(\alpha)$ is also known as the di-gamma function.

The solution of (2.1) can easily be obtained using the Newton-Raphson method with $\hat{\alpha}_M$ as the starting value for $\hat{\alpha}_L$.

## 2.3 Method of Product Spacings (MPS)

The method of product spacings (MPS) was concurrently introduced by Cheng and Amin (1983) and Ranneby (1984). Let

$$D_i = \int_{x_{i-1:n}}^{x_{i:n}} f(x;\theta)dx, \quad i = 1, 2, \ldots, n+1,$$

where $x_{0:n}$ is the lower limit and $x_{n+1:n}$ is the upper limit of the domain of the density function $f(x;\theta)$, and $\theta$ can be vector-valued. Also, $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are defined as an ordered random sample from $f(x;\theta)$. Clearly, the spacings sum to unity, that is $\sum D_i = 1$. The MPS method is, quite simply, to choose $\theta$ to maximize the geometric mean of the spacings,

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

or, equivalently, its logarithm

$$H = \ln G.$$

MPS estimation gives consistent estimators under much more general conditions than MLEs. MPS estimators are asymptotically normal and are asymptotically as efficient as MLEs when these exist. For detailed goodness properties of MPS estimators, readers are referred to Cheng and Amin (1983), Ranneby (1984), Cheng and Iles (1987), Shah and Gokhale (1993), Rahman and Pearson (2002) and the references therein. Using the density function (1.1) and the cdf (1.2), H can be written as follows:

$$\begin{aligned} H &= \frac{1}{n+1}\left[\ln F(X_{1:n};\beta,\alpha) + \ln\left\{1 - F(X_{n:n};\beta,\alpha)\right\}\right] \\ &\quad + \frac{1}{n+1}\left[\sum_{i=1}^{n-1} \ln\left\{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)\right\}\right] \end{aligned} \tag{2.2}$$

By maximizing (2.2) for different values of $\beta$ and $\alpha$, the MPS estimates can be obtained as $\hat{\beta}_P$ and $\hat{\alpha}_P$. The Newton-Raphson method can be used in solving when the two first derivatives are equal to zero. The MME's are used as the starting values. The first derivatives of $H$ with respect to $\beta$ and $\alpha$ are respectively,

$$\begin{aligned} H'_\beta &= \frac{1}{n+1}\left[\frac{F'_\beta(X_{1:n};\beta,\alpha)}{F(X_{1:n};\beta,\alpha)} + \sum_{i=1}^{n-1} \frac{F'_\beta(X_{i+1:n};\beta,\alpha) - F'_b(X_{i:n};\beta,\alpha)}{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)}\right. \\ &\quad \left. - \frac{F'_\beta(X_{n:n};\beta,\alpha)}{1 - F(X_{n:n};\beta,\alpha)}\right] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} H'_\alpha &= \frac{1}{n+1}\left[\frac{F'_\alpha(X_{1:n};\beta,\alpha)}{F(X_{1:n};\beta,\alpha)} + \sum_{i=1}^{n-1} \frac{F'_\alpha(X_{i+1:n};\beta,\alpha) - F'_\alpha(X_{i:n};\beta,\alpha)}{F(X_{i+1:n};\beta,\alpha) - F(X_{i:n};\beta,\alpha)}\right. \\ &\quad \left. - \frac{F'_\alpha(X_{n:n};\beta,\alpha)}{1 - F(X_{n:n};\beta,\alpha)}\right] \end{aligned} \tag{2.4}$$

where

$$
\begin{aligned}
F'_\beta(x;\beta,\alpha) &= \frac{\alpha}{\beta}\left[F(x;\beta,\alpha+1)-F(x;\beta,\alpha)\right], \\
F'_\alpha(x;\beta,\alpha) &= E_x(\ln x;\beta,\alpha)-F(x;\beta,\alpha)(\ln\beta+\Psi(\alpha)), \\
E_x(\ln x;\beta,\alpha) &= \int_0^x \ln t\frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}dt.
\end{aligned}
$$

The second derivatives of $H$ with respect to $\beta$ and $\alpha$ are respectively,

$$
\begin{aligned}
H''_{\beta\beta} = \frac{1}{n+1}\Bigg[ & \frac{F(X_{1:n};\beta,\alpha)F''_{\beta\beta}(X_{1:n};\beta,\alpha)-\left\{F'_\beta(X_{1:n};\beta,\alpha)\right\}^2}{\{F(X_{1:n};\beta,\alpha)\}^2} \\
& +\sum_{i=1}^{n-1}\frac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\beta\beta}(X_{i+1:n};\beta,\alpha)-F''_{\beta\beta}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\left\{F'_\beta(X_{i+1:n};\beta,\alpha)-F'_\beta(X_{i:n};\beta,\alpha)\right\}^2}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\beta\beta}(X_{n:n};\beta,\alpha)+\left\{F'_\beta(X_{n:n};\beta,\alpha)\right\}^2}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Bigg],
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
H''_{\beta\alpha} = \frac{1}{n+1}\Bigg[ & \frac{F(X_{1:n};\beta,\alpha)F''_{\beta\alpha}(X_{1:n};\beta,\alpha)-F'_\beta(X_{1:n};\beta,\alpha)F'_\alpha(X_{1:n};\beta,\alpha)}{\{F(X_{1:n};\beta,\alpha)\}^2} \\
& +\sum_{i=1}^{n-1}\frac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\beta\alpha}(X_{i+1:n};\beta,\alpha)-F''_{\beta\alpha}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\left\{F'_\beta(X_{i+1:n};\beta,\alpha)-F'_\beta(X_{i:n};\beta,\alpha)\right\}\left\{F'_\alpha(X_{i+1:n};\beta,\alpha)-F'_\alpha(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\beta\alpha}(X_{n:n};\beta,\alpha)+F'_\beta(X_{n:n};\beta,\alpha)F'_\alpha(X_{n:n};\beta,\alpha)}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Bigg]
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
H''_{\alpha\alpha} = \frac{1}{n+1}\Bigg[ & \frac{F(X_{1:n};\beta,\alpha)F''_{\alpha\alpha}(X_{1:n};\beta,\alpha)-\left\{F'_\alpha(X_{1:n};\beta,\alpha)\right\}^2}{\{F(X_{1:n};\beta,\alpha)\}^2} \\
& +\sum_{i=1}^{n-1}\frac{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}\left\{F''_{\alpha\alpha}(X_{i+1:n};\beta,\alpha)-F''_{\alpha\alpha}(X_{i:n};\beta,\alpha)\right\}}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\left\{F'_\alpha(X_{i+1:n};\beta,\alpha)-F'_\alpha(X_{i:n};\beta,\alpha)\right\}^2}{\{F(X_{i+1:n};\beta,\alpha)-F(X_{i:n};\beta,\alpha)\}^2} \\
& -\frac{\{1-F(X_{n:n};\beta,\alpha)\}F''_{\alpha\alpha}(X_{n:n};\beta,\alpha)+\left\{F'_\alpha(X_{n:n};\beta,\alpha)\right\}^2}{\{1-F(X_{n:n};\beta,\alpha)\}^2}\Bigg],
\end{aligned}
\tag{2.7}
$$

where

$$
F''_{\beta\beta}(x;\beta,\alpha)=\frac{\alpha(\alpha+1)}{\beta^2}\left[F(x;\beta,\alpha+2)-2F(x;\beta,\alpha+1)+F(x;\beta,\alpha)\right],
$$

$$
\begin{aligned}
F''_{\beta\alpha}(x;\beta,\alpha) &= \tfrac{\alpha}{\beta}\left[E_x(\ln x;\beta,\alpha+1)-E_x(\ln x;\beta,\alpha)\right] \\
&\quad -\tfrac{\alpha}{\beta}F(x;\beta,\alpha+1)(\ln\beta+\Psi(\alpha)-\tfrac{1}{\beta}F(x;\beta,\alpha)(1-\alpha\ln\beta-\alpha\Psi(\alpha),
\end{aligned}
$$

$$
\begin{aligned}
F''_{\alpha\alpha}(x;\beta,\alpha) &= E_x((\ln x)^2;\beta,\alpha)-2E_x(\ln x;\beta,\alpha)(\ln\beta+\Psi(\alpha)) \\
&\quad +F(x;\beta,\alpha)\left[(\ln\beta)^2+2\ln\beta\Psi(\alpha)-\Psi'(\alpha)+\Psi(\alpha)\right],
\end{aligned}
$$

with $\Psi'(\alpha)$ being the derivative of $\Psi(\alpha)$, and

$$E_x((\ln x)^2;\beta,\alpha) = \int_0^x (\ln t)^2 \frac{t^{\alpha-1}e^{-t/\beta}}{\Gamma(\alpha)\beta^\alpha}dt.$$

Then, the multivariate Newton-Raphson iteration is performed as

$$\begin{bmatrix} \hat{\beta}_P^{(l+1)} \\ \hat{\alpha}_P^{(l+1)} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_P^{(l)} \\ \hat{\alpha}_P^{(l)} \end{bmatrix} - \begin{bmatrix} H''^{(l)}_{\beta\beta} & H''^{(l)}_{\beta\alpha} \\ H''^{(l)}_{\beta\alpha} & H''^{(l)}_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} H'^{(l)}_{\beta} \\ H'^{(l)}_{\alpha} \end{bmatrix}, \tag{2.8}$$

where $l$ is the index for the iterations, and $\hat{\alpha}_P$ and $\hat{\beta}_P$ are resulting product spacing estimates for $\alpha$ and $\beta$, respectively.

## 2.4 Quantile Estimates (QE)

Methods of estimation which are based on using the quantiles of the corresponding distributions are denoted as Quantile Estimates (QE). Recently, Schmid (1997) considered percentile estimators for the three-parameter weibull distribution and Castillo and Hadi (1995) considered the quantiles of continuous random variables in estimating their parameters. Readers are referred to these two references and the references there in for historical background and for other details. Quantile estimates (QE) in general can be summarized as follows:

Let $\theta = \{\theta_1, \theta_2, \ldots, \theta_r\}$ be the parameters to be estimated and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics obtained from a random sample from $F(x;\theta)$, where, for fixed $\theta$, $F(x;\theta)$ is assumed to be strictly increasing on the interior of its support. Also, let $I = \{i_1, i_2, \ldots, i_r\}$ be a set of $r$ distinct indices, where $i_j \in \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, r\}$. Then, one can write

$$F(x_{i:n};\theta) \cong p_{i:n}, \quad i \in I$$

or, equivalently,

$$x_{i:n} \cong F^{-1}(p_{i:n};\theta), \quad i \in I, \tag{2.9}$$

where $p_{i:n} = (i-a)/(n+b)$ is an empirical distribution of $F(x_{i:n};\theta)$ or suitable plotting positions, and $a$ and $b$ are constants. The values of $a$ and $b$ are chosen (either theoretically or based on simulation) so that the resulting estimators have certain desirable properties (e.g., the minimum root mean square error). Replacing the approximation by equality in (2.9), we get a set of $r$ independent equations in $r$ unknowns, $\theta_1, \theta_2, \ldots, \theta_r$. An elemental estimate of $\theta$ can then be obtained by solving (2.9) for $\theta$. Note that these elemental estimates are based on the percentile method.

The estimates obtained from (2.9) depend on $r$ observations. A subset of size $r$ observations is known as an elemental subset and the resultant estimate is known as an elemental estimate of $\theta$. Thus, from a sample of size $n$, there are $nCr$ elemental estimates. For large $n$ and $r$, the number of elemental subsets may be too large for the computations of all elemental estimates to be feasible. In such cases, instead of computing all possible elemental estimates, one may select a pre specified number, $N$, of elemental subsets either systemically, based on some theoretical considerations, or at random. For each of these subsets, an elemental estimate of $\theta$ is computed and is denoted as $\hat{\theta}_{j1}, \hat{\theta}_{j2}, \ldots, \hat{\theta}_{jN}$, $j = 1, 2, \ldots, r$. These elemental estimates can then be combined, using some suitable (preferably robust) functions, to obtain an overall final estimate of $\theta$. A commonly used robust function is the median (MED). That is,

$$\hat{\theta} = \text{median}(\hat{\theta}_{j1}, \hat{\theta}_{j2}, \ldots, \hat{\theta}_{jN}).$$

The estimates are unique even when the method of moments (MOM) and the MLE equations have multiple solutions or when the MOM and the MLE do not exist.

### 2.4.1 Two-parameter Gamma Distribution

In the two-parameter gamma distribution (1.1), the cdf in (1.2) is $F(x; \alpha, \beta)$ from which it follows that the $p$th quantile is

$$q(p; \alpha, \beta) = F^{-1}(p; \alpha, \beta); \quad 0 < p < 1.$$

There are two parameters, so two equations are needed. Let $I = \{i, j\}$ so that (2.9) becomes

$$\begin{aligned} x_{i:n} &= F^{-1}(p_{i:n}; \alpha, \beta), \\ x_{j:n} &= F^{-1}(p_{j:n}; \alpha, \beta), \end{aligned}$$

where $i < j$, from which the elemental estimates of $\alpha$ and $\beta$ are obtained and denoted as $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$. Then for $p_{i:n}$, $i = 1, 2, \ldots, n$, overall estimates for $\alpha$ and $\beta$ are obtained as

$$\hat{\alpha}_Q = \text{median}(\hat{\alpha}_{ij}) \quad \text{and} \quad \hat{\beta}_Q = \text{median}(\hat{\beta}_{ij}),$$

where $Q$ stands for a quantile estimate.

In simulations in Section 5, the empirical quantiles $p_{i:n} = i/(n+1)$ are used.

## 3 Simulation Results

One thousand samples are generated for two different parameter settings

$$\{(\beta = 0.5, \alpha = 0.5) \text{ and } (\beta = 2.0, \alpha = 4.0)\}$$

and for three different sample sizes ($n = 10$, $n = 25$, and $n = 50$). Means (MEAN), standard deviations (SD), biases (BIAS), mean of the absolute biases (MAB) and mean squared errors (MSE) are computed and displayed in Table 1.

MATLAB software is used in all computations and the codes are readily available.

Table 1: Simulation Results

| | $\hat{\beta}_M$ | $\hat{\beta}_L$ | $\hat{\beta}_P$ | $\hat{\beta}_Q$ | $\hat{\alpha}_M$ | $\hat{\alpha}_L$ | $\hat{\alpha}_P$ | $\hat{\alpha}_Q$ |
|---|---|---|---|---|---|---|---|---|
| $\beta = 0.5$ $\alpha = 0.5$ $n = 10$ | | | | | | | | |
| MEAN | 0.3836 | 0.5154 | 0.6824 | 0.5794 | 0.8609 | 0.6410 | 0.5193 | 0.5729 |
| SD | 0.2779 | 1.2250 | 0.4928 | 0.3817 | 0.4899 | 0.3180 | 0.2462 | 0.3312 |
| BIAS | -0.1164 | 0.0154 | 0.1824 | 0.0794 | 0.3609 | 0.1410 | 0.0193 | 0.0729 |
| MAB | 0.2562 | 0.2811 | 0.3384 | 0.2726 | 0.4143 | 0.2169 | 0.1756 | 0.2008 |
| MSE | 0.0908 | 1.5008 | 0.2761 | 0.1520 | 0.3703 | 0.1210 | 0.0610 | 0.1150 |
| $\beta = 0.5$ $\alpha = 0.5$ $n = 25$ | | | | | | | | |
| MEAN | 0.4477 | 0.5256 | 0.6408 | 0.5500 | 0.6539 | 0.5556 | 0.4734 | 0.5140 |
| SD | 0.2355 | 1.4681 | 0.2880 | 0.2060 | 0.2476 | 0.1387 | 0.1198 | 0.1357 |
| BIAS | -0.0523 | 0.0256 | 0.1408 | 0.0500 | 0.1539 | 0.0556 | -0.0266 | 0.0140 |
| MAB | 0.1852 | 0.1848 | 0.2121 | 0.1591 | 0.2151 | 0.1045 | 0.0969 | 0.1009 |
| MSE | 0.0582 | 2.1560 | 0.1028 | 0.0449 | 0.0850 | 0.0223 | 0.0151 | 0.0186 |
| $\beta = 0.5$ $\alpha = 0.5$ $n = 50$ | | | | | | | | |
| MEAN | 0.4650 | 0.4708 | 0.5726 | 0.5139 | 0.5880 | 0.5446 | 0.4830 | 0.5134 |
| SD | 0.1788 | 0.1171 | 0.1867 | 0.1347 | 0.1694 | 0.0826 | 0.0897 | 0.0836 |
| BIAS | -0.0350 | -0.0292 | 0.0726 | 0.0139 | 0.0880 | 0.0446 | -0.0170 | 0.0134 |
| MAB | 0.1410 | 0.0980 | 0.1331 | 0.1059 | 0.1457 | 0.0670 | 0.0701 | 0.0613 |
| MSE | 0.0332 | 0.0146 | 0.0401 | 0.0183 | 0.0364 | 0.0088 | 0.0083 | 0.0072 |
| $\beta = 2.0$ $\alpha = 4.0$ $n = 10$ | | | | | | | | |
| MEAN | 1.7047 | 1.7715 | 2.8055 | 2.3296 | 5.9528 | 5.6770 | 3.7181 | 4.5786 |
| SD | 0.8395 | 0.8307 | 1.3260 | 1.2242 | 3.7543 | 3.6137 | 2.3682 | 3.1575 |
| BIAS | -0.2953 | -0.2285 | 0.8055 | 0.3296 | 1.9528 | 1.6770 | -0.2819 | 0.5786 |
| MAB | 2.3211 | 2.2494 | 1.5337 | 1.8611 | 3.9617 | 3.6819 | 1.7929 | 2.6219 |
| MSE | 0.7919 | 0.7423 | 2.4071 | 1.6073 | 17.9087 | 15.8709 | 5.6879 | 10.3046 |
| $\beta = 2.0$ $\alpha = 4.0$ $n = 25$ | | | | | | | | |
| MEAN | 1.9224 | 1.9444 | 2.4629 | 2.2074 | 4.6298 | 4.5109 | 3.6308 | 4.0736 |
| SD | 0.6284 | 0.5733 | 0.7304 | 0.7150 | 1.5392 | 1.4116 | 1.1224 | 1.3917 |
| BIAS | -0.0776 | -0.0556 | 0.4629 | 0.2074 | 0.6298 | 0.5109 | -0.3692 | 0.0736 |
| MAB | 2.0807 | 2.0557 | 1.5603 | 1.8041 | 2.6308 | 2.5109 | 1.6330 | 2.0752 |
| MSE | 0.4009 | 0.3317 | 0.7477 | 0.5543 | 2.7658 | 2.2536 | 1.3960 | 1.9423 |
| $\beta = 2.0$ $\alpha = 4.0$ $n = 50$ | | | | | | | | |
| MEAN | 1.9565 | 1.9728 | 2.2719 | 2.1099 | 4.3215 | 4.2418 | 3.7261 | 4.0163 |
| SD | 0.4639 | 0.4168 | 0.4824 | 0.4836 | 0.9894 | 0.8740 | 0.7624 | 0.9087 |
| BIAS | -0.0435 | -0.0272 | 0.2719 | 0.1099 | 0.3215 | 0.2418 | -0.2739 | 0.0163 |
| MAB | 2.0437 | 2.0272 | 1.7286 | 1.8904 | 2.3215 | 2.2418 | 1.7261 | 2.0163 |
| MSE | 0.2171 | 0.1745 | 0.3066 | 0.2459 | 1.0823 | 0.8223 | 0.6562 | 0.8260 |

Table 2: Failure Times

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 620 | 470 | 260 | 89 | 388 | 242 | 103 | 100 | 39 | 460 | 284 |
| 1285 | 218 | 393 | 106 | 158 | 152 | 477 | 403 | 103 | 69 | 158 |
| 818 | 947 | 399 | 1274 | 32 | 12 | 134 | 660 | 548 | 381 | 203 |
| 871 | 193 | 531 | 317 | 85 | 1410 | 250 | 41 | 1101 | 32 | 421 |
| 32 | 343 | 376 | 1512 | 1792 | 47 | 95 | 76 | 515 | 72 | 1585 |
| 253 | 6 | 860 | 89 | 1055 | 537 | 101 | 385 | 176 | 11 | 565 |
| 164 | 16 | 1267 | 352 | 160 | 195 | 1279 | 356 | 751 | 500 | 803 |
| 560 | 151 | 24 | 689 | 1119 | 1733 | 2194 | 763 | 555 | 14 | 45 |
| 776 | 1 | | | | | | | | | |

# 4 Application

The following data in Table 2 represents failure times of machine parts from manufacturer A and are taken from `http://v8doc.sas.com/sashtml/stat/chap29/sect44.htm`

For this data, $\hat{\beta}_M = 483.22$, $\hat{\beta}_L = 550.60$, $\hat{\beta}_P = 604.13$, $\hat{\beta}_Q = 610.7$, $\hat{\alpha}_M = 0.97$, $\hat{\alpha}_L = 0.85$, $\hat{\alpha}_P = 0.80$, and $\hat{\alpha}_Q = 0.79$.

# 5 Summary and Concluding Remarks

From Table 1, it is observed that all the estimates appear to be consistent and asymptotically unbiased. In estimating the scale parameter $\beta$, the maximum likelihood estimate is better in all aspects except in terms of the mean absolute bias (MAB) the product spacings performed better for larger samples. In estimating the shape parameter $\alpha$, the method of product spacings performed better in all aspects except biases are smaller for the quantile estimates. The standard errors are smaller for larger samples in case of quantile estimates.

# References


[1] Kloke, J.D, McKean, J.W., and Salman, M. (2005). A small sample investigation of rank–based profile analyses. *Journal of Statistical Research*, **39**(2), 155–182.

[2] Atkinson, A.C., Donev, A.N., and Tobias, R.D. (2007). *Optimum experimental designs, with SAS*. Oxford University Press.

[3] Castillo, E. and Hadi, A.S. (1995). A method for estimating parameters and quantiles of distributions of continuous random variables. *Computational Statistics & Data Analysis*, **20**, 421–439.

[4] Cheng, R.C.H. and Amin, N.A.K. (1983). Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin. *Journal of Royal Statistical Society, Series B*, **45**(3), 394–403.

[5] Cheng, R.C.H. and Iles, T.C. (1987). Corrected Maximum Likelihood in Non-Regular Problems. *Journal of Royal Statistical Society, Series B*, **49**(1), 95–101.

[6] Choi, S.C. and Wette, R. (1969). Maximum Likelihood Estimation of the Parameters of the Gamma Distribution and Their Bias. *Technometrics*, **11**(4), 683–690.

[7] Dang, H. and Weerakkody, G. (2000). Bounds for the Maximum Likelihood Estimates in Two-Parameter Gamma Distribution. *Journal of Mathematical Analysis and Applications*, **245**(1), 1–6.

[8] Evans, M., Hastings, N. and Peacock, B. (2000). *Statistical Distributions*, Third Edition. John Wiley & Sons, Inc., New York.

[9] Harter, H.L. and Moore, A.H. (1965). Maximum likelihood estimation of the parameters of Gamma and Weibull populations from complete and from censored samples. *Technometrics*, **7**(4), 639–643.

[10] Lee, E. T. (1992). *Statistical Methods for Survival Data Analysis*, Second Edition. John Wiley & Sons, Inc., New York.

[11] Rahman, M., Pearson, L.M., and Martinovic, U.R. (2007). Method of Product Spacings in the Two-Parameter Gamma Distributions. *Journal of Statistical Research*, **41**(1), 51–58.

[12] Rahman, M. and Pearson, L.M. (2002). Estimation in Two-Parameter Exponential Distributions. *Journal of Statistical Computation and Simulation*, **70**(4), 371–386.

[13] Ranneby, B. (1984). The Maximum Spacing Method. An Estimation Method Related to the Maximum Likelihood Method. *Scandinavian Journal of Statistics*, **11**, 93–112.

[14] Schmidt, U. (1997). Percentile estimators for the three-parameter weibull distribution for use when all parameters are unknown. *Communications in Statistics — Theory and Methods*, **26**(3), 765–785.

[15] Shah, A. and Gokhale, D.V. (1993). On Maximum Product of Spacings (MPS) Estimation for Burr XII Distributions. *Communications in Statistics — Simulation and Computation*, **22**(3), 615–641.

[16] Wilks, D.S. (1990). Maximum Likelihood Estimation for the Gamma Distribution Using Data Containing Zeros. *Journal of Climate*, **3**, 1495–1501.