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SOME CHARACTERIZATIONS OF EXPONENTIAL DISTRIBUTION BY RECORD VALUES

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SUMMARY

Suppose $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed absolutely continuous random variables and $X_{U(1)} = X_1,\ X_{U(2)}, \ldots$ be the upper records from the sequence. In this paper we will consider several distributional properties of the upper records from the exponential distribution. Based on these distributional properties some characterizations of the exponential distribution are presented.

Keywords and phrases: Records, Characterization, Exponential Distribution

1 Introduction

Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous distribution function F and the corresponding probability density function f. We define $U(1) = 1, U(n+1) = min\{j|j > U(n),\ X_j > X_{U(n)}\}$. The sequence $\{X_{U(n)}\}(\{U(n)\})$ is known as upper record values (record times).

If $f_n(x)$ is the pdf of $X_{U(n)}$, then

$$f_n(x) = \frac{(R(x))^{n-1}}{\Gamma(n)} f(x),$$

where $R(x) = -ln(1 - F(x)), 0 < F(x) < 1$. It can easily be shown that

$$F_n(x) = e^{-R(x)} \sum_{j=0}^{n-1} \frac{(R(x))^j}{\Gamma(j+1)}. \tag{1.1}$$

The joint pdf $f_{n,m}(x,y)$ of $X_{U(n)}$ and $X_{U(m),}$ $1 \le m < n$, can be written as

$$f_{m,n}(x,y) = \frac{(R(x))^{m-1}}{\Gamma(m)\Gamma(n-m)}[R(y) - R(x)]^{n-m-1} r(x) f(y), \quad -\infty < x < y < \infty.$$

Let X be a random variable (rv) whose probability density function (pdf) is given by

$$f(x,\theta) = \begin{cases} \theta e^{-\theta x}, & x \ge 0,\ \theta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

We say $X \in E(\theta)$ if the pdf of X is as given in (1.1).In this paper we will discuss several distributional properties of the upper record values when $X_i \in E(\theta)$. We will present some characterizations of the exponential distribution based on the distributional properties of the upper records.

2 Main Results

If $X_i \in E(\theta)$, $i = 1, \ldots,$ then it can be shown (see, Ahsanullah (2004))

$$X_{U(n)} \stackrel{d}{=} W_1 + W_2 + \cdots + W_n, \tag{2.1}$$

where $W_1, W_2, \ldots, W_n$ are i.i.d with the pdf as given in (1.1). From (2.1), it follows that

$$Z_i = X_{U(i)} - X_{U(i-1)},\ i = 2, \ldots, n$$

with $X_{U(0)} = 0$ are identically distributed as exponential with pdf as given in (1.1).

If $X_i \in E(\theta)$, then pdf of $X_{U(n)}, n \ge 1, x, y,$ is

$$f_n(x) = \frac{\theta^n x^{n-1}}{\Gamma(n)} e^{-\theta x}.$$

The joint pdf of $X_{U(n)}$ and $X_{U(m)},\ 1 \le m < n$ is

$$f_{m,n}(x,y) = \frac{\theta^n x^{m-1}}{\Gamma(m)} \frac{(y-x)^{n-m-1}}{\Gamma(n-m)} e^{-\theta y},\ 0 < x < y < \infty.$$

The conditional pdf of $X_{U(n)}|X_{U(m)}$ is

$$f_{n|m}(y|x) = \frac{\theta^{n-m}(y-x)^{n-m-1}}{\Gamma(n-m)} e^{-\theta(y-x)},\ 0 < x < y < \infty.$$

For characterizations of exponential distribution using equality of distributions of $Z_{n+1,n} = X_{U(n+1)} - X_{U(n)}$ and X_k , $k \ge 1$ see Ahsanullah 2004, p. 81. There are several characterizations of the exponential distribution using the regression properties of record values, see Nagaraja (1977), Ahsanullah and Wesolowski (1998), Dembinska and Wesolowski (2000).

The exponential distribution can be characterized by the equality in distribution of $X_{U(n)}$ and $X_{U(n-1)} + X_{U(1)}$, (see the Lemma below).

If F is a distribution function of a non-negative random variable,we will call F is "new better than used (NBU) if $1 - F(x+y) \leq (1-F(x))(1-F(y))$, and F is "worse than used (NWU) if $1 - F(x+y) \geq (1-F(x))(1-F(y)),\ x, y \geq 0$. We will say $F \in C$ if F is either NBU or NWU.

Lemma 2.1. *Let $X_1, X_2, \ldots,$ be a sequence of non-negative i.i.d. random variables with cdf $F(x)$ and pdf $f(x)$. We assume $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then the following two statements are equivalent*

(a) $X \in E(\theta)$ for some $\theta > 0$

(b) $X_{U(n)} \stackrel{d}{=} X_{U(n-1)} + W$, where $F \in C$, W is independent of $X_{U(n-1)}$ and is distributed as F for any fixed $n > 1$.

Proof. It is easy to show that (a)$\Longrightarrow$(b). We will proof here (b)$\Longrightarrow$(a). We have

$$\begin{aligned} F_n(x) &= \int_0^x P(X \leq x-y) f_{n-1}(y)dy \\ &= \int_0^x [1-(1-F(x-y))] f_{n-1}(y)dy \\ &= F_{n-1}(x) - \int_0^x [1-(1-F(x-y))] f_{n-1}(y)dy \end{aligned}$$

i.e.

$$\begin{aligned} F_{n-1}(x) - F_n(x) &= \int_0^x (1-F(x-y))] f_{n-1}(y)dy \\ &\geq \int_0^x \frac{1-F(x)}{1-F(y)} f_{n-1}(y)dy \text{ if } F \text{ is NBU.} \end{aligned}$$

Thus,

$$\frac{F_{n-1}(x) - F_n(x)}{1-F(x)} \geq \int_0^x \frac{f_{n-1}(y)}{1-F(y)} dy = \int_0^x \frac{R^{n-2}(y)}{\Gamma(n-1)} r(y)dy = \frac{R^{n-1}(x)}{\Gamma(n)} \tag{2.2}$$

where $r(y) = \frac{f(y)}{1-F(y)}$.

But by (1.1), we have

$$\frac{F_{n-1}(x) - F_n(x)}{1-F(x)} = \frac{(R(x))^{n-1}}{\Gamma(n)}.$$

Thus (2.2) is a contradiction if F is NBU unless

$$1 - F(x-y) = (1-F(x))/(1-F(y) \tag{2.3}$$

for almost all $x,\ y,\ 0 < y \le x < \infty$.

The solution of (2.3) with the boundary condition $F(0) = 0$ is

$$F(x) = 1 - e^{-\theta x}, \tag{2.4}$$

where $x \ge 0,\ \theta > 0$.

If F is NWU, then we get the same solution (2.4). Using (2.1), we see that if $X_i \in E(\theta)$, then

$$X_{U(n)} \stackrel{d}{=} X_{U(m)} + G(n-m), 1 \le m < n,$$

where $G(r)$ is the gamma distribution with pdf $f_1(x)$ as

$$f_1(x) = \frac{\theta^r}{\Gamma(r)} x^{r-1} e^{-\theta x}, x \ge 0.$$

The following theorem gives a solution for $m = n-2$. □

Theorem 1. *Let $X_1, X_2, \ldots,$ be a sequence of non-negative i.i.d. random variables with cdf $F(x)$ and twice differentiable pdf $f(x)$. We assume $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then the following two statements are equivalent*

(a) $X \in E(\theta)$ for some $\theta > 0$.

(b) $X_{U(n)} \stackrel{d}{=} X_{U(n-2)} + W_{n,m}$ where $W_{n,m}$ is independent of $X_{U(n-2)}$ and has pdf

$$f_1(y) = \frac{\theta^{n-m} y^{n-m-1}}{\Gamma(n-m)} e^{-\theta y},\ \ y \ge 0,\ \ \textit{for}\ m = n-2.$$

Proof. (a)$\Longrightarrow$(b) is trivial, so we will proof here (b)$\Longrightarrow$(a). Let's first derive $P(W_{n.m} \le t)$. Integrating by parts and repeating the integration we have

$$\begin{aligned} P(W_{n.m} \le t) &= \int_0^t \frac{\theta^{n-m} y^{n-m-1}}{\Gamma(n-m)} e^{-\theta y} dy \\ &= 1 - e^{-\theta t}[1 + \theta t + \theta^2 t^2/2! + \cdots + \frac{\theta^{n-m-1} t^{n-m-1}}{(n-m-1)!}]. \end{aligned}$$

By (b) we have

$$\begin{aligned} F_n(x) &= \int_0^x P(W_{n,m} \le x-y) f_m(y) dy \\ &= \int_0^x [1 - e^{-\theta(x-y)}(1 + \theta(x-y) + \cdots + \frac{\theta^{n-m-1}(x-y)^{n-m-1}}{(n-m-1)!})] f_m(y) dy \end{aligned}$$

or

$$e^{\theta x}[F_m(x) - F_n(x)] = \int_0^x e^{\theta y}[1 + \theta(x-y) + \cdots + \frac{\theta^{n-m-1}(x-y)^{n-m-1}}{(n-m-1)!}]f_m(y)dy. \quad (2.5)$$

Differentiating left hand side

$$\begin{aligned}\frac{d}{dx}\{e^{\theta x}[F_m(x) - F_n(x)]\} &= \frac{d}{dx}\{e^{\theta x}(1-F(x))\sum_{k=m}^{n-1} R^k(x)/k!\} \\ &= e^{\theta x} f(x) \sum_{k=m}^{n-1} \frac{R^{k-1}(x)}{(k-1)!} + \sum_{k=m}^{n-1} \frac{R^k(x)}{k!}\frac{d}{dx}e^{\theta x - R(x)}. \quad (2.6) \\ &= e^{\theta x} f(x) \sum_{k=m}^{n-1} \frac{R^{k-1}(x)}{(k-1)!} + \sum_{k=m}^{n-1} \frac{R^k(x)}{k!} e^{\theta x - R(x)}(\theta - R'(x)).\end{aligned}$$

Using the equality

$$\frac{d}{dx}\int\limits_0^x e^{\theta y}\theta^s \frac{(x-y)^s}{s!} f_m(y)dy = \int\limits_0^x e^{\theta y}\theta^s \frac{(x-y)^{s-1}}{(s-1)!} f_m(y)dy,$$

differentiating the right hand side of (2.5) and combining with (2.6) we have

$$\begin{aligned}& e^{\theta x} f(x) \sum_{k=m+1}^{n-1} \frac{R^{k-1}(x)}{(k-1)!} + \sum_{k=m}^{n-1} \frac{R^k(x)}{k!} e^{\theta x - R}(\theta - R'(x)) \\ =\ & \int_0^x e^{\theta y}\left[\theta + \theta^2(x-y) + \cdots + \frac{\theta^{n-m-1}(x-y)^{n-m-2}}{(n-m-2)!}\right] f_m(y)dy\end{aligned}$$

Taking into account $m = n-2$ and $e^{\theta x} f(x) = e^{\theta x - R(x)} R'(x)$ in the last equation gives

$$\frac{R^m(x)}{m!} e^{\theta x - R(x)} R'(x) + \left(\frac{R^m(x)}{m!} + \frac{R^{m+1}(x)}{(m+1)!}\right) e^{\theta x - R(x)}(\theta - R'(x)) = \theta \int_0^x e^{\theta y} f_m(y)dy,$$

i.e.

$$\theta\left(\frac{R^m(x)}{m!} + \frac{R^{m+1}(x)}{(m+1)!}\right) e^{\theta x - R(x)} - \frac{R^{m+1}(x)}{(m+1)!} e^{\theta x - R(x)} R'(x) = \theta \int_0^x e^{\theta y} f_m(y)dy. \quad (2.7)$$

Differentiating both sides of (2.7) w.r.t. x, we obtain

$$\begin{aligned}& \theta\left(\frac{R^m(x)}{m!} + \frac{R^{m+1}(x)}{(m+1)!}\right) e^{\theta x - R(x)}(\theta - R'(x)) + \theta\left(\frac{R^{m-1}(x)}{(m-1)!} + \frac{R^m(x)}{m!}\right) e^{\theta x - R(x)} R'(x) \\ & -\frac{R^m(x)}{m!} e^{\theta x - R(x)}[R'(x)]^2 - \frac{R^{m+1}(x)}{(m+1)!}[e^{\theta x - R(x)}(\theta - R'(x))R'(x) + e^{\theta x - R(x)} R''(x)] \\ & \quad = \theta \frac{R^{m-1}(x)}{(m-1)!} e^{\theta x - R(x)} R'(x).\end{aligned}$$

On simplification, we obtain

$$(\theta - R'(x))[\theta + R'(x) + (\theta - R'(x))R(x)/(m+1)] = R''(x)R(x)/(m+1). \tag{2.8}$$

Now let's show that (2.8) has the only solution $R(x) = \theta x$, or, equivalently $R'(x) = \theta$ for all $x \geq 0$. Note that $R''(x)$ exists for all x and

$$R''(x) = \frac{f'(x)(1 - F(x)) + f^2(x)}{(1 - F(x))^2}.$$

Taking limit on both sides of (2.8) when $x \to 0$ we have $(\theta - R'(0))(\theta + R'(0)) = 0$. So, it follows that $R'(0) = \theta$.

Let's show that if $R'(x_0) = \theta$ for any $x_0 \geq 0$, then $R'(x)$ cannot change in the right neighbor of x_0. If $R'(x) > \theta$ ($R'(x) < \theta$) and x is close to x_0 the first parenthesis of (2.8) is negative(positive) and the second parenthesis is always positive because

$$\theta + R' + (\theta - R')R/(m+1) \sim \theta + R' > 0.$$

Hence R'' is negative(positive). Since $R'(x_0) = \theta$ and R'' is negative(positive) $R'(x)$ is decreasing (increasing) and must be less(more) than θ. It contradicts $R'(x) > \theta$ ($R'(x) < \theta$). Hence $R'(x) = \theta$ for all $x \geq 0$. It means that $F(x) = 1 - e^{-\theta x},\ x \geq 0$ for some $\theta > 0$.

This proves the Theorem. □

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