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ESTIMATION OF THE MEAN VECTOR OF A MULTIVARIATE NORMAL MODEL UNDER REFLECTED NORMAL LOSS

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summary

For estimating an unknown mean vector-parameter $\boldsymbol{\theta}$ in a multivariate normal population, we propose the unrestricted, restricted and preliminary test estimators and derive their exact risk expressions under a modified reflected normal loss function. This approach is an extension to the work of Giles [2002. Preliminary-Test and Bayes Estimation of A Location Parameter Under Reflected Normal Loss, in Ullah, A. and Chaturvedi, A, Handbook of Applied Econometrics and Statistical Inference, Marcel Decker, New York, 287-303]. Comparison are then made for more clarity of the behavior of the estimators.

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1 Introduction

The properties of an estimator primarily depends on the chosen loss function; and the choice of loss function depends on the objectives of the study. For example, it is well known that the ordinary least squares estimators under certain standard assumptions are the best linear unbiased. However, if the objective of any study is to minimize some specific risk function then other types of estimators may perform better than the ordinary least squares estimator. In recent years there has been growing interest to estimate the parameters under different loss functions (cf. Saleh, 2006).

Many authors have considered quadratic loss function for evaluating the risk functions of underlying estimators. Among others the elegant work of Saleh (2006) has systematically treated some improved estimators in different standard statistical models considering quadratic loss function. Also some other popular loss functions (using absolute-error loss, asymmetric linear exponential (LINEX) loss, or balanced loss) include the contributions of Ohtani et al.(1997), Giles et al.(1996), Ohtani and Giles (1996), Giles and Giles (1996), and more recently Arashi et al. (2008), among others. Despite its tractability and historical interest, two obvious practical shortcomings of the quadratic loss function are its unboundedness and its symmetry (Giles, 2002). Recently, Spring (1993) has addressed the issue of unboundedness, by analyzing the reflected normal (RN) loss function, and has motivated it through an example in quality assurance (e.g., Taguchi, 1986). The reflected normal loss function has the practical merit that it is bounded. It can readily be made asymmetric if desired. Giles (2002) considered RN loss function in evaluating the risk functions of maximum likelihood (ML), restricted ML (RML) and preliminary test (PT) estimators of a scalar mean in a univariate normal distribution. The objective of this paper is to propose ML, RML and PT estimators based on a sample from a multivariate normal population and and to provide the dominance order of the estimators based on the RN loss.
More precisely, this paper involves the problem of estimation of the mean vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$ of a multivariate normal model when it is suspected that $\boldsymbol{\theta}$ may belong to the sub-space defined by $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ where $\boldsymbol{\theta}_0$ is a p-vector of known pre-specified values with focus on the PT estimator of $\boldsymbol{\theta}$.

Let $\boldsymbol{\theta}$ be a p-vector parameter to be estimated, and $\boldsymbol{\delta}$ be a statistic used as an estimator of $\boldsymbol{\theta}$, then we define the modified reflected normal loss function as

$$L(\boldsymbol{\delta}; \boldsymbol{\theta}) = K\left\{1 - \exp\left[-\frac{(\boldsymbol{\delta}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\delta}-\boldsymbol{\theta})}{2\gamma^2}\right]\right\}, \tag{1.1}$$

where K is the maximum loss and γ is a pre-assigned shape parameter that controls the rate at which the loss approaches its upper bound. The reflected normal loss structure in the context of M-estimation (e.g. Huber, 1977), and in the context of robust estimation its influence function is known to have rather good properties. The graph of RN loss comparing with that of quadratic is given in Figure 1.

2 Model and proposed estimators

Let $X_1, ..., X_N$ be independent and identically distributed (iid) as $N_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ where the mean vector $\boldsymbol{\theta}$ and positive definite (p.d.) covariance matrix $\boldsymbol{\Sigma}$ are both unknown.

It is well-known that the MLE of $\boldsymbol{\theta}$ is given by

$$\tilde{\boldsymbol{\theta}} = \bar{\boldsymbol{X}}, \tag{2.1}$$

where $\bar{\boldsymbol{X}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{X}_i$, is an unbiased estimator of mean vector $\boldsymbol{\theta}$ say, and is an unrestricted estimator. Also the corresponding unbiased estimator of $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{S} = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})'. \tag{2.2}$$

If we know that when hypothesis $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ holds, then the restricted estimator is

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0. \tag{2.3}$$

In practice, the prior information that $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0$, is uncertain. The uncertainty in the prior knowledge can be removed by using Bancroft (1944) estimator, considering test of the hypothesis $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against the alternative $H_A : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. As a result of this test we choose $\tilde{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\theta}}$ based on the rejection or acceptance of H_0.

Accordingly, we write the PT estimator as

$$\hat{\boldsymbol{\theta}}^{PT} = \hat{\boldsymbol{\theta}} I(U < F_{p,m}(\alpha)) + \tilde{\boldsymbol{\theta}} I(U \geq F_{p,m}(\alpha)), \tag{2.4}$$

where $F_{p,m}(\alpha)$ is the α-level upper critical value of a central F-distribution with $(p, m = N - p)$ degrees of freedom, $I(A)$ is the indicator function of the set A and U is the test statistic for testing the null hypothesis $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ (or $\boldsymbol{\delta} = \boldsymbol{\theta} - \boldsymbol{\theta}_0 = \mathbf{0}$), against $H_A : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$.

Direct computations concludes that U has the following form

$$\begin{aligned} U &= \frac{mN}{np} (\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0)' \boldsymbol{S}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0) \\ &= \frac{m}{np} T^2, \qquad n = N - 1, \end{aligned} \tag{2.5}$$

where

$$T^2 = N(\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0)' \boldsymbol{S}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0). \tag{2.6}$$

The test statistic U follows a central F-distribution with (p, m) degrees of freedom under H_0 while under the alternative it follows the non-central F-distribution with (p, m) degrees of freedom and non-centrality parameter $\frac{\Delta^2}{2}$, where

$$\begin{aligned} \Delta^2 &= N(\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\theta} - \boldsymbol{\theta}_0) \\ &= N \boldsymbol{\delta}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\delta}. \end{aligned} \tag{2.7}$$

The preliminary test approach estimation has been pioneered by Bancroft (1944), Saleh and Sen (1978), Kibria and Saleh (1993). The performance of PTE depends on the size of the test $(0 < \alpha < 1)$, and the choice of estimators remains between the two values $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$. Depending on the outcome of the test, it yields the extreme results, namely $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$.

In the forthcoming section, we evaluate the bias and risk functions of the proposed estimators under reflected normal loss.

3 Bias and Risk Expressions

Suppose $\boldsymbol{\theta}^*$ is an estimator of $\boldsymbol{\theta}$. In this section we derive the bias and risk functions respectively expressed by the notations $\boldsymbol{b}(\boldsymbol{\theta}^*)$, and $\boldsymbol{R}(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ for each ML, RML and PT estimator considered in Section 2 given by

$$\begin{aligned} \boldsymbol{b}(\boldsymbol{\theta}^*) &= E(\boldsymbol{\theta}^* - \boldsymbol{\theta}), \\ \boldsymbol{R}(\boldsymbol{\theta}^*;\boldsymbol{\theta}) &= E[L(\boldsymbol{\theta}^*;\boldsymbol{\theta})], \end{aligned}$$

where $L(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ is given by (1.1).

Theorem 1. *The bias vectors of the ML, RML and PT estimators of $\boldsymbol{\theta}$ are given by*

(i) $\boldsymbol{b}_1(\tilde{\boldsymbol{\theta}}) = 0$,

(ii) $\boldsymbol{b}_2(\hat{\boldsymbol{\theta}}) = -\boldsymbol{\delta}, \qquad \boldsymbol{\delta} = \boldsymbol{\theta} - \boldsymbol{\theta}_0$,

(iii) $\boldsymbol{b}_3(\hat{\boldsymbol{\theta}}^{PT}) = -\boldsymbol{\delta}\mathcal{G}_{p+2,m}\left(\frac{p}{p+2}F_{p,m}(\alpha);\Delta^2\right)$,

where $\mathcal{G}_{p,m}\left(.;\Delta^2\right)$ is the cdf of non-central F-distribution with (p,m) degrees of freedom and non-centrality parameter $\frac{\Delta^2}{2}$ with

$$\mathcal{G}_{p,m}\left(x;\Delta^2\right) = \sum_{r\geq 0} \frac{e^{\frac{\Delta^2}{2}}}{r!}\left(\frac{\Delta^2}{2}\right)^r I_y\left(\frac{1}{2}p+r;\frac{1}{2}m\right), \quad y = \frac{px}{m+px},$$

where $I_y(a,b)$ is an incomplete beta function.

Proof: The expression of $\boldsymbol{b}_1(\tilde{\boldsymbol{\theta}})$ is obvious. For $\boldsymbol{b}_2(\hat{\boldsymbol{\theta}})$, using equation (2.3) we have

$$\boldsymbol{b}_2(\hat{\boldsymbol{\theta}}) = E(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = E(\boldsymbol{\theta}_0 - \boldsymbol{\theta}) = -\boldsymbol{\delta}.$$

Next by using (2.4) we get

$$\begin{aligned} \boldsymbol{b}_3(\hat{\boldsymbol{\theta}}^{PT}) &= E(\hat{\boldsymbol{\theta}}^{PT} - \boldsymbol{\theta}) \\ &= E\left[\bar{\boldsymbol{X}} - (\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0)I\left(U < F_{p,m}(\alpha)\right) - \boldsymbol{\theta}\right] \\ &= -E\left[(\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0)I\left(U < F_{p,m}(\alpha)\right)\right] \\ &= -\frac{\boldsymbol{\Sigma}^{\frac{1}{2}}}{N^{\frac{1}{2}}}E\left[\boldsymbol{Z}\times I\left(\frac{T^2}{n} < \frac{p}{m}F_{p,m}(\alpha)\right)\right], \end{aligned}$$

where, $\boldsymbol{Z} = N^{\frac{1}{2}}\boldsymbol{\Sigma}^{\frac{-1}{2}}(\bar{\boldsymbol{X}} - \boldsymbol{\theta}_0) \sim N_p(\boldsymbol{\eta}, \boldsymbol{I}_p)$ and

$$\boldsymbol{\eta} = N^{\frac{1}{2}}\boldsymbol{\Sigma}^{\frac{-1}{2}}(\boldsymbol{\theta} - \boldsymbol{\theta}_0) = N^{\frac{1}{2}}\boldsymbol{\Sigma}^{\frac{-1}{2}}\boldsymbol{\delta}. \tag{3.1}$$

Note that $\Delta^2 = \boldsymbol{\eta}'\boldsymbol{\eta}$. Also, from Theorem 3.2.13 of Muirhead (2005) we have

$$\frac{T^2}{n} = \frac{\boldsymbol{Z}'\boldsymbol{Z}}{\chi^2_m},$$

where $\boldsymbol{Z}'\boldsymbol{Z} \sim \chi^2_p(\Delta^2)$ is independently distributed of χ^2_m. Hence, from Lemma 2 of Judge and Bock (1978 p.320), one can obtain the bias vector of PTE as

$$\begin{aligned}
\boldsymbol{b}_3(\hat{\boldsymbol{\theta}}^{PT}) &= -\frac{\boldsymbol{\Sigma}^{\frac{1}{2}}}{N^{\frac{1}{2}}} E\left[\boldsymbol{Z} \times I\left(\frac{\boldsymbol{Z}'\boldsymbol{Z}}{\chi^2_m} < \frac{p}{m} F_{p,m}(\alpha)\right)\right] \\
&= -N^{-\frac{1}{2}} \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\eta}\, E\left[I\left(\frac{\chi^2_{p+2}(\Delta^2)}{\chi^2_m} < \frac{p}{m} F_{p,m}(\alpha)\right)\right] \\
&= -\boldsymbol{\delta} P\left[F_{p+2,m}(\Delta^2) < \frac{p}{p+2} F_{p,m}(\alpha)\right] \qquad \text{by (3.1)} \\
&= -\boldsymbol{\delta} \mathcal{G}_{p+2,m}\left(\frac{p}{p+2} F_{p,m}(\alpha); \Delta^2\right).
\end{aligned}$$

Note that for $\alpha = 0$, the bias vector of $\hat{\boldsymbol{\theta}}^{PT}$ coincides with the bias vector of the restricted estimator, $\hat{\boldsymbol{\theta}}$, while for $\alpha = 1$, it coincides with that of $\tilde{\boldsymbol{\theta}}$, the unrestricted estimator. Also, under $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$, all the proposed estimators are unbiased since $\boldsymbol{\delta} = \boldsymbol{0}$.

In the following theorem we state the risk functions of ML, RML and PT estimators.

Theorem 2. *The risks of the ML, RML and PT estimators of $\boldsymbol{\theta}$ under the reflected normal loss function are given by*

(i) $\boldsymbol{R}(\tilde{\boldsymbol{\theta}}; \boldsymbol{\theta}) = K\left(1 - \left(1 + \frac{1}{N\gamma^2}\right)^{-\frac{p}{2}}\right)$,

(ii) $\boldsymbol{R}(\hat{\boldsymbol{\theta}}; \boldsymbol{\theta}) = K\left\{1 - \exp\left[-\frac{\Delta^2}{2N\gamma^2}\right]\right\}$,

(iii) $\boldsymbol{R}(\hat{\boldsymbol{\theta}}^{PT}; \boldsymbol{\theta}) = K\Bigg\{1 - \sum_{r=0}^{\infty} \frac{1}{(-1)^r (2\gamma^2)^r r!}\Bigg(\left(\frac{2}{N}\right)^r \frac{\Gamma\left(\frac{p}{2}+r\right)}{\Gamma\left(\frac{p}{2}\right)} + \left(\frac{\Delta^2}{N}\right)^r Pr.\left(\frac{\chi^2_p(\Delta^2)}{\chi^2_m} < c^*_\alpha\right)$

$-\sum_{j=0}^{r}\left[\binom{r}{j}\left(\frac{\Delta^2}{N}\right)^j \left(\frac{2}{N}\right)^{r-j} \sum_{i=0}^{\infty} \frac{e^{-\Delta^2}(\Delta^2)^i}{i!} \frac{\Gamma\left(\frac{p}{2}+r-j+i\right)}{\Gamma\left(\frac{p}{2}+i\right)} Pr.\left(\frac{\chi^2_{p+2r-2j+2i}(\Delta^2)}{\chi^2_m} < c^*_\alpha\right)\right]\Bigg)\Bigg\}$,

where c_α is critical value for a chosen significance level α and $c^*_\alpha = \frac{p}{m}\, c_\alpha$.

Proof: Using equation (1.1)

$$\begin{aligned}
\boldsymbol{R}(\tilde{\boldsymbol{\theta}}; \boldsymbol{\theta}) &= KE\left\{1 - \exp\left[-\frac{(\bar{\boldsymbol{X}} - \boldsymbol{\theta})' \boldsymbol{\Sigma}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{\theta})}{2\gamma^2}\right]\right\} \\
&= KE\left\{1 - \exp\left[-\frac{y}{2N\gamma^2}\right]\right\},
\end{aligned}$$

where $y = N(\bar{\boldsymbol{X}} - \boldsymbol{\theta})' \boldsymbol{\Sigma}^{-1} (\bar{\boldsymbol{X}} - \boldsymbol{\theta}) \sim \chi^2_p$. So by making use of the moment generating function of the chi-square with p d.f., we obtain

$$\boldsymbol{R}(\tilde{\boldsymbol{\theta}}; \boldsymbol{\theta}) = K\left(1 - \left(1 + \frac{1}{N\gamma^2}\right)^{-\frac{p}{2}}\right).$$

Afterward, the risk of UE is trivial, as $\boldsymbol{\theta}_0$ is a constant, and can be evaluated as

$$\begin{aligned}\boldsymbol{R}(\hat{\boldsymbol{\theta}};\boldsymbol{\theta}) &= KE\left\{1-\exp\left[-\frac{(\boldsymbol{\theta}_0-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}_0-\boldsymbol{\theta})}{2\gamma^2}\right]\right\} \\ &= K\left\{1-\exp\left[-\frac{\Delta^2}{2N\gamma^2}\right]\right\}.\end{aligned}$$

Finally, recall that

$$\begin{aligned}\hat{\boldsymbol{\theta}}^{PT} &= \hat{\boldsymbol{\theta}}I\left(U<F_{p,m}(\alpha)\right)+\tilde{\boldsymbol{\theta}}I\left(U\geq F_{p,m}(\alpha)\right) \\ &= \boldsymbol{\theta}_0 I_A(U)+\bar{\mathbf{X}}I_R(U),\end{aligned}$$

we have

$$\begin{aligned}\boldsymbol{R}(\hat{\boldsymbol{\theta}}^{PT};\boldsymbol{\theta}) &= K\left\{1-E\left(\exp\left[-\frac{(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})}{2\gamma^2}\right]\right)\right\} \\ &= K\left\{1-\sum_{r=0}^{\infty}\frac{E\left[(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})\right]^r}{(-1)^r(2\gamma^2)^r r!}\right\}.\end{aligned}$$

Also note that

$$\begin{aligned}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta}) &= [\boldsymbol{\theta}_0 I_A(U)+\bar{\mathbf{X}}I_R(U)-\boldsymbol{\theta}]'\boldsymbol{\Sigma}^{-1}[\boldsymbol{\theta}_0 I_A(U)+\bar{\mathbf{X}}I_R(U)-\boldsymbol{\theta}] \\ &= I_R(U)(\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})+(\boldsymbol{\theta}-\boldsymbol{\theta}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_0)I_A(U).\end{aligned}$$

Using the fact that $I_R(U)\times I_A(U)=0$, for each $r\in\{0,1,\cdots\}$ we have

$$\begin{aligned}\left[(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})\right]^r &= \frac{1}{N^r}\left[N(\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})I_R(U)\right]^r \\ &\quad +\left[(\boldsymbol{\theta}-\boldsymbol{\theta}_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_0)I_A(U)\right]^r \\ &= \frac{1}{N^r}\left[N(\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right]^r \\ &\quad +\left[\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^r-\left((\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right)^r\right]I_A(U).\end{aligned}$$

Then, we obtain

$$\begin{aligned}R(\hat{\boldsymbol{\theta}}^{PT};\boldsymbol{\theta}) = K\Bigg\{1-\sum_{r=0}^{\infty}\Bigg\{&\frac{1}{(-1)^r(2\gamma^2)^r r!}\frac{1}{N^r}E\left[N(\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right]^r \\ &+E\left\{\left[\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^r-\left((\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right)^r\right]I_A(U)\right\}\Bigg\}\Bigg\}.\end{aligned} \tag{3.2}$$

Now, from the moments of the chi-square distribution with p d.f. we have

$$E\left[N(\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right]^{r}=\frac{2^{r}\Gamma(\frac{p}{2}+r)}{\Gamma(\frac{p}{2})}. \tag{3.3}$$

Also,

$$\begin{aligned}
&E\left\{\left[\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^{r}-\left((\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right)^{r}\right]I_A(U)\right\}\\
=\ &E\left\{\left[\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^{r}-\sum_{j=0}^{r}\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^{j}\left[(\bar{\mathbf{X}}-\boldsymbol{\theta}_0)'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta}_0)\right]^{r-j}\binom{r}{j}\right]I_A(U)\right\}\\
=\ &E\left\{\left[\left(\frac{\Delta^2}{N}\right)^{r}-\sum_{j=0}^{r}\left(\frac{\Delta^2}{N}\right)^{j}\left(\frac{\chi_p^2(\Delta^2)}{N}\right)^{r-j}\binom{r}{j}\right]I_A(U)\right\}. \qquad (3.4)
\end{aligned}$$

Recalling that $U=\frac{m}{p}\frac{\chi_p^2(\Delta^2)}{\chi_m^2}$, where the two chi-square variates in the numerator and denominator are independent, we can express (3.4) as:

$$\begin{aligned}
E\left\{\left[\left(\delta'\boldsymbol{\Sigma}^{-1}\delta\right)^{r}-\left((\bar{\mathbf{X}}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\theta})\right)^{r}\right]I_A(U)\right\}&=\left(\frac{\Delta^2}{N}\right)^{r}Pr.\left(\frac{\chi_p^2(\Delta^2)}{\chi_m^2}<c_\alpha^*\right)\\
&-\sum_{j=0}^{r}\left\{\left(\frac{\Delta^2}{N}\right)^{j}N^{j-r}\binom{r}{j}E\left[I_A\left(\frac{m\chi_p^2(\Delta^2)}{p\chi_m^2)}\right)\left(\chi_p^2(\Delta^2)\right)^{r-j}\right]\right\}. \qquad (3.5)
\end{aligned}$$

The expectation in (3.5) can be evaluated by using the result of Clarke (1986, Appendix 1) repeatedly. Therefore we obtain

$$\begin{aligned}
E\left[I_A\left(\frac{m\chi_p^2(\Delta^2)}{p\chi_m^2)}\right)\left(\chi_p^2(\Delta^2)\right)^{r-j}\right]&=2^{r-j}\sum_{i=0}^{\infty}\frac{e^{-\Delta^2}(\Delta^2)^i}{i!}\frac{\Gamma\left(\frac{p}{2}+r-j+i\right)}{\Gamma\left(\frac{p}{2}+i\right)}\\
&\times Pr.\left(\frac{\chi_{p+2r-2j+2i}^2(\Delta^2)}{\chi_m^2}<c_\alpha^*\right). \qquad (3.6)
\end{aligned}$$

Finally, by equations (3.3)-(3.6), we can write the risk of the PTE, as

$$\begin{aligned}
\boldsymbol{R}(\hat{\boldsymbol{\theta}}^{PT};\boldsymbol{\theta})=K\Bigg\{1-\sum_{r=0}^{\infty}\frac{1}{(-1)^r(2\gamma^2)^r r!}\Bigg\{&\left(\frac{2}{N}\right)^{r}\frac{\Gamma\left(\frac{p}{2}+r\right)}{\Gamma\left(\frac{p}{2}\right)}+\left(\frac{\Delta^2}{N}\right)^{r}Pr.\left(\frac{\chi_p^2(\Delta^2)}{\chi_m^2}<c_\alpha^*\right)\\
&-\sum_{j=0}^{r}\Bigg\{\binom{r}{j}\left(\frac{\Delta^2}{N}\right)^{j}\left(\frac{2}{N}\right)^{r-j}\sum_{i=0}^{\infty}\frac{e^{-\Delta^2}(\Delta^2)^i}{i!}\frac{\Gamma\left(\frac{p}{2}+r-j+i\right)}{\Gamma\left(\frac{p}{2}+i\right)}\\
&\times Pr.\left(\frac{\chi_{p+2r-2j+2i}^2(\Delta^2)}{\chi_m^2}<c_\alpha^*\right)\Bigg\}\Bigg\}\Bigg\}.
\end{aligned}$$

4 Analysis of Risks

In this section, we provide the risk analysis of the estimators under reflected normal loss.

First, we compare $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$. In general, the risk difference is given by

$$\boldsymbol{R}(\tilde{\boldsymbol{\theta}};\boldsymbol{\theta}) - \boldsymbol{R}(\hat{\boldsymbol{\theta}};\boldsymbol{\theta}) = \exp\left[-\frac{\Delta^2}{2N\gamma^2}\right] - \left(1+\frac{1}{N\gamma^2}\right)^{\frac{-p}{2}}.$$

Thus one can see that the risk difference is non-positive ,i.e., $\tilde{\boldsymbol{\theta}}$ performs better than $\hat{\boldsymbol{\theta}}$ denoted by ($\tilde{\boldsymbol{\theta}} \succcurlyeq \hat{\boldsymbol{\theta}}$) provided

$$\Delta^2 > pN\gamma^2 \ln\left(1+\frac{1}{N\gamma^2}\right),$$

while $\hat{\boldsymbol{\theta}}$ performs better than $\tilde{\boldsymbol{\theta}}$ whenever

$$\Delta^2 < pN\gamma^2 \ln\left(1+\frac{1}{N\gamma^2}\right).$$

Under H_0, $\boldsymbol{R}(\hat{\boldsymbol{\theta}};\boldsymbol{\theta}) = 0$. So under H_0, $\hat{\boldsymbol{\theta}}$ performs better than $\tilde{\boldsymbol{\theta}}$.
In general, the risk functions for the restricted and unrestricted estimators under reflected normal loss are easily evaluated for particular choices of the parameters and sample size, and these are illustrated in figure (2). In particular, we see there that $\boldsymbol{R}(\hat{\boldsymbol{\theta}};\boldsymbol{\theta})$ is bounded above by K.
In the comparison of $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ with $\hat{\boldsymbol{\theta}}^{PT}$, $\hat{\boldsymbol{\theta}}^{PT}$ performs better than $\tilde{\boldsymbol{\theta}}$ under risk difference whenever

$$\begin{aligned}\sum_{r=0}^{\infty}\frac{1}{(-1)^r(2\gamma^2)^r r!}\Bigg\{&\left(\frac{2}{N}\right)^r\frac{\Gamma\left(\frac{p}{2}+r\right)}{\Gamma\left(\frac{p}{2}\right)} + \left(\frac{\Delta^2}{N}\right)^r Pr.\left(\frac{\chi_p^2(\Delta^2)}{\chi_m^2}<c_\alpha^*\right)\\ &-\sum_{j=0}^{r}\Bigg\{\binom{r}{j}\left(\frac{\Delta^2}{N}\right)^j\left(\frac{2}{N}\right)^{r-j}\sum_{i=0}^{\infty}\frac{e^{-\Delta^2}(\Delta^2)^i}{i!}\frac{\Gamma\left(\frac{p}{2}+r-j+i\right)}{\Gamma\left(\frac{p}{2}+i\right)}\\ &Pr.\left(\frac{\chi^2_{p+2r-2j+2i}(\Delta^2)}{\chi_m^2}<c_\alpha^*\right)\Bigg\}\Bigg\} \geq \left(1+\frac{1}{N\gamma^2}\right)^{-\frac{p}{2}},\end{aligned}$$

otherwise $\hat{\boldsymbol{\theta}} \succcurlyeq \hat{\boldsymbol{\theta}}^{PT}$.

The evaluation of the risk of the preliminary-test estimator is rather more tedious, but it can be readily verified through numerical computations. Some examples of this appear in figure (3) and (4). In particular, figure (3) compares $\boldsymbol{R}(\hat{\boldsymbol{\theta}}^{PT};\boldsymbol{\theta})$ with $\boldsymbol{R}(\hat{\boldsymbol{\theta}};\boldsymbol{\theta})$ and $\boldsymbol{R}(\tilde{\boldsymbol{\theta}};\boldsymbol{\theta})$ for a sample size N=10, and illustrative parameter values. For the values of $0.095 < \Delta^2$ of the parameter space, $\hat{\boldsymbol{\theta}}^{PT}$ with $\alpha = 0.05$, is least preferred among the three estimators under consideration. Similarly, for the values of $\Delta^2 < 0.095$, $\hat{\boldsymbol{\theta}}$ dominates $\tilde{\boldsymbol{\theta}}$ and for the values of parameter space under which $\Delta^2 > 0.095$, $\tilde{\boldsymbol{\theta}}$ performs petter that $\hat{\boldsymbol{\theta}}$. There are regions where either $\hat{\boldsymbol{\theta}}$ or $\tilde{\boldsymbol{\theta}}$ are most preferred among the three estimators, but there is no region of the parameter space where the pre-test estimator is preferred over both $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ simultaneously. The effect of increasing the significance level for the preliminary test from 0.05 to 0.15 can be seen in figure (4). For example for the values of $0 < \Delta^2 < 0.109$ of the parameter space, $\hat{\boldsymbol{\theta}}^{PT}$ with $\alpha = 0.15$, is most preferred among the three estimators under consideration and increasing the level of significance improve the performance of preliminary test estimators among the others. At last, for the case $p = 1$, the obtained results are the same as those in Giles (2002).

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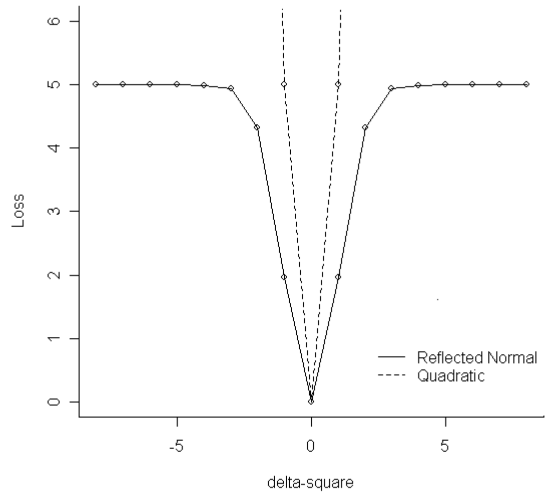

Figure 1: Comparison of SEL and RNL (N=10,p=4,gamma=1,K=1)

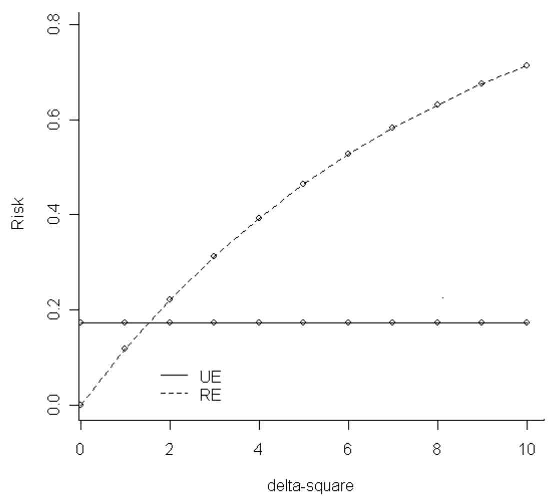

Figure 2: Risk under reflected normal loss (N=10,p=4,gamma=1,K=1)

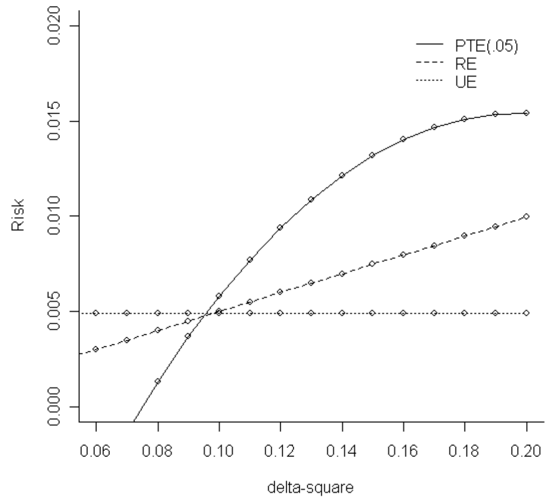

Figure 3: Risk under reflected normal loss (N=10,p=4,m=6,gamma=1,K=1)

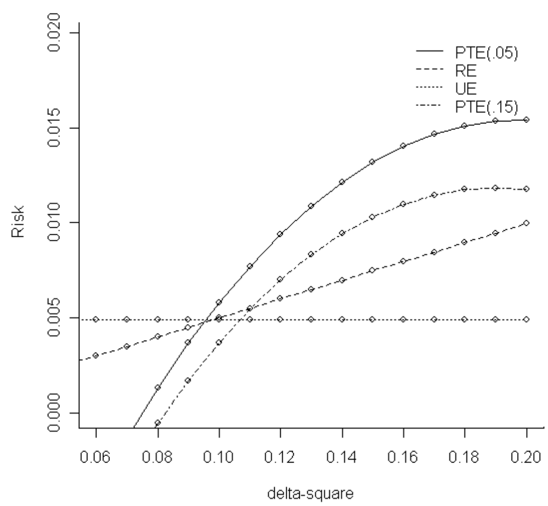

Figure 4: Risk under reflected normal loss(N=10,p=4,m=6,gamma=1,K=1)