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IMPROVED ESTIMATION OF THE PARAMETERS OF SIMPLE LINEAR REGRESSION MODEL WITH AUTOCORRELATED ERRORS

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SUMMARY

The problem of estimating the parameters of a simple linear regression model with known autocorrelated errors is the main concern of this paper. We consider the unrestricted estimator, restricted estimator, preliminary test estimator (PTE) and shrinkage estimator (SE) of the parameters θ and β. The bias and MSE expressions of the proposed estimators are given and analyzed. From the study of some graphs and efficiency tables, it is evident that the proposed preliminary test and shrinkage estimators dominate the unrestricted estimator for some values of Δ. Sample tables of maximum/minimum efficiencies are provided for various ρ and level of significance values to assess the performance of the estimators and choose optimum levels of significance of the PTE. For small size of the test, the SE performs better than the PT estimator, while for larger sizes, the PT performs better than the SE.

Keywords and phrases: Autocorrelation; Dominance; Efficiency; MSE; Pre-Test; Shrinkage Estimation.

AMS Classification: 62J07, 62F03

1 Introduction

The linear regression model has significant applications in various fields of sciences. Parameters estimation for the regression model is a major concern for practitioners. Several researchers have proposed different estimation techniques since its inception. It is usual practice that the regression parameters are to be estimated by using sample information. In many practical situations, the researcher begins the analysis not only with the sample information but also with some prior knowledge of the data. If the prior knowledge about the parameters is correct, the inclusion of the prior information in the estimation process would improve the quality of the estimators. The prior information may be known from theory or from the past experiments. The convex combination of the sample and prior information results the pre-test estimator. The preliminary test estimation approach has been proposed by Bancroft (1944). Since 1944 several researchers have used this procedure for various purposes. Among many, Bancroft (1964), Han and Bancroft (1968), Judge and Bock (1978), Saleh and Kibria (1993), Benda (1996), Chiou (1997), Ahmed and Saleh (1990, 1999), Han (2002), and Kibria and Saleh (2005, 2006) are notable. A detailed review of the preliminary test estimation procedures are given by Han at al. (1988) and Giles and Giles (1993) among others. Most recently, Saleh (2006) broadened the scope of PTE technique under parametric, nonparametric and asymptotic setup. Saleh, in a preliminary note introduced the idea of shrinkage estimators (SE) for univariate problems as a competitor of PTE which culminated in the contributions with Khan and Saleh (2001) and Khan, Hoque and Saleh (2002). Details of this procedure are available in Saleh (2006) with applications in measurement error model in "Kim and Saleh (2003a,b)".

The observations for the linear model are usually assumed to be independent. However, for many practical situations the observations are autocorrelated instead of independent. Simple linear regression with autocorrelated errors has a fundamental role in many statistical analysis (for examples, see Harvey and Phillips, 1979; Zinde-Walsh and Galbraith, 1991; Choudhury et al., 1999). Different types of estimators for the regression coefficients of the linear regression model with autocorrelated disturbances have been proposed in the literature. The usual estimation techniques are generalized, weighted and least squares estimators (Grenander 1954, Lee and Lund 2004). Beside these estimators, in the previous studies, researchers frequently attempted to deal with the problem of auto regression in disturbances by using the method of first differences (for example, Kadiyala (1968), Maeshiro (1976, 1980) and Kramer (1983)).

The literature on PTE in a linear model with auto-correlated errors is limited to Saleh (2006) and Judge and Bock (1978). This paper gives a full study of the properties of PTE and SE of intercept and slope parameters when ρ is known. The organization of this paper is as follows. The model and proposed estimators are given in section 2. The bias and MSE expressions of the proposed estimators are given in section 3. A comparison has been made in section 4. Relative efficiency is discussed in section 5. Finally some concluding remarks are given in section 6.

2 Model and Proposed Estimators

Consider the following simple regression model with autocorrelated errors given by

$$Y = \theta 1_n + \beta x + \epsilon_n, \tag{2.1}$$

where $\epsilon_n = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)'$ and satisfies the first-order stationery autocorrelated model defined by

$$\epsilon_i = \rho\epsilon_{i-1} + v_i, \quad |\rho| < 1; \quad i = 1, 2, \dots, n \tag{2.2}$$

with

$$E(v) = 0, \quad E(vv') = \sigma_v^2 I_n,$$

where $v = (v_1, v_2, \dots, v_n)'$. Now, ϵ_n has a Gaussian distribution with

$$E(\epsilon_n) = 0, \quad E(\epsilon_n \epsilon_n') = \sigma_v^2 \Sigma_\rho,$$

where

$$\Sigma_\rho = \begin{pmatrix} 1 & \rho & \dots & \rho^{n-1} \\ \rho & 1 & \dots & \rho^{n-2} \\ \dots & \dots & \dots & \dots \\ \rho^{n-1} & \rho^{n-2} & \dots & 1 \end{pmatrix} \tag{2.3}$$

with the inverse

$$\Sigma_\rho^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & 0 & \dots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -\rho & 1 \end{pmatrix}. \tag{2.4}$$

Now from (2.1), it is clear that $Y \sim N(\theta 1_n + \beta x, \sigma_v^2 \Sigma_\rho)$.

2.1 Unrestricted Estimator of β and θ

When $\rho = 0$, the usual unbiased estimators of θ and β are given by

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} \bar{Y} - \tilde{\beta}\bar{x} \\ \frac{1}{Q}[x'Y - \frac{1}{n}(1_n'x)(1_n'Y)] \end{pmatrix} \tag{2.5}$$

with covariance matrix

$$\frac{\sigma_v^2}{n} \begin{pmatrix} 1 + \frac{n\bar{x}}{Q} & -\frac{n\bar{x}}{Q} \\ -\frac{n\bar{x}}{Q} & \frac{1}{Q} \end{pmatrix}, \tag{2.6}$$

where $Q = x'x - \frac{1}{n}(1_n'x)^2$. The unbiased estimator of σ_v^2 is

$$s_e^2 = \frac{1}{n-2}(Y - \tilde{\theta}1_n - \tilde{\beta}x)'(Y - \tilde{\theta}1_n - \tilde{\beta}x). \tag{2.7}$$

Now for the model (2.1), the unbiased estimators of θ and β can be obtained based on the following generalized least squares estimation (GLSE) procedure as

$$\begin{pmatrix} \tilde{\theta}(\rho) \\ \tilde{\beta}(\rho) \end{pmatrix} = \frac{1}{K_1K_2 - K_3^2} \begin{pmatrix} K_2 1_n'\Sigma_\rho^{-1}Y - K_3 x'\Sigma_\rho^{-1}Y \\ K_1 x'\Sigma_\rho^{-1}Y - K_3 1_n'\Sigma_\rho^{-1}Y \end{pmatrix} \tag{2.8}$$

with the covariance matrix

$$cov \begin{pmatrix} \tilde{\theta}(\rho) \\ \tilde{\beta}(\rho) \end{pmatrix} = \frac{\sigma_v^2}{K_1K_2 - K_3^2} \begin{pmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{pmatrix} = \sigma_v^2 (X'\Sigma_\rho^{-1}X)^{-1}, \tag{2.9}$$

where $X = (1_n', x')'$ and

$$K_1 = (1_n'\Sigma_\rho^{-1}1_n) = \frac{n}{1+\rho}\left(1 - \frac{\rho}{n}\right)$$

$$K_2 = x'\Sigma_\rho^{-1}x = \frac{1}{1-\rho^2}\left\{(1-\rho^2)x_1^2 + \sum_{i=2}^{n}(x_i - \rho x_{i-1})^2\right\}$$

and

$$K_3 = 1_n'\Sigma_\rho^{-1}x = \frac{n}{1+\rho}\left[\bar{x} - \frac{\rho}{n}\left(\sum_{i=1}^{n-1} x_i\right)\right]$$

are functions of autocorrelation, ρ assumed to be known. For notational convenience, here we suppressed ρ in the quantities K_1, K_2 and K_3.

To test the null hypothesis $H_0 : \beta = 0$ against $H_A : \beta \neq 0$, we use the following test statistic,

$$\mathcal{L} = \frac{[\tilde{\beta}(\rho)]^2}{\tilde{\sigma}_v^2 K_1}[K_1K_2 - K_3^2].$$

Under H_0, $\mathcal{L}$ has $F_{1,m}$ distribution and under H_A, it has non-central F distribution with degrees of freedom (1, m) and non-centrality parameter

$$\Delta = \frac{\beta^2[K_1K_2 - K_3^2]}{\sigma_v^2 K_1},$$

where

$$\sigma_v^2 = \frac{1}{n-2}[Y_n - \tilde{\theta}(\rho)1_n - \tilde{\beta}(\rho)x]'\Sigma_\rho^{-1}[Y_n - \tilde{\theta}(\rho)1_n - \tilde{\beta}(\rho)x].$$

If $\beta = 0$ with certainly, the restricted estimator of θ can be written as

$$\tilde{\theta}(\rho) = K_1^{-1}[1_n'\Sigma_\rho^{-1}Y_n r] = \tilde{\theta}(\rho) + \frac{K_3}{K_1}\tilde{\beta}(\rho) \tag{2.10}$$

with variance

$$Var(\tilde{\theta}(\rho) = \frac{\sigma_v^2}{K_1}.$$

2.2 Preliminary Test and S-Estimation of β and θ

Assume that $F_{1,m}(\alpha)$ is α-level critical value of $\mathcal{L}_n$ under H_0, then the PTE of β and θ are define by

$$\hat{\beta}^{PT}(\rho) = \tilde{\beta}(\rho) - \tilde{\beta}(\rho) I(\mathcal{L}_n < F_{1,m}(\alpha)) \tag{2.11}$$

and

$$\begin{aligned} \hat{\theta}^{PT}(\rho) &= \tilde{\theta}(\rho) - [\tilde{\theta}(\rho) - \hat{\theta}(\rho)] I(\mathcal{L}_n < F_{1,m}(\alpha)) \\ &= \tilde{\theta}(\rho) + \frac{K_3}{K_1} \tilde{\beta} I(\mathcal{L}_n < F_{1,m}(\alpha)), \end{aligned} \tag{2.12}$$

respectively. Similarly we may define the S (shrinkage) -estimators of β and θ as

$$\hat{\beta}^S(\rho) = \left\{ 1 - \frac{c\tilde{\sigma}_v K_1^{1/2}}{(K_1 K_2 - K_3^2)^{1/2} |\tilde{\beta}(\rho)|} \right\} \tilde{\beta}(\rho) \tag{2.13}$$

and

$$\hat{\theta}^S(\rho) = \tilde{\theta}(\rho) + \frac{c\tilde{\sigma}_v K_3 \tilde{\beta}(\rho)}{K_1^{1/2} (K_1 K_2 - K_3^2)^{1/2} |\tilde{\beta}(\rho)|}, \tag{2.14}$$

respectively.

3 Bias and MSE Expressions

3.1 Bias and MSE Expressions of β

Since the restricted estimator of β is 0 and that of θ is $\hat{\theta}$ (since $\beta = 0$), following Saleh (2006), the bias and mean square error (MSE) of the estimators of β are given below:

(i) Unrestricted estimator (UE):

$$\begin{aligned} b_1(\tilde{\beta}(\rho)) &= 0 \\ M_1(\tilde{\beta}(\rho)) &= \frac{\sigma_v^2 K_1}{K_1 K_2 - K_3^2}. \end{aligned} \tag{3.1}$$

(ii) Preliminary test (PT) estimator:

$$\begin{aligned} b_2(\hat{\beta}^{PT}(\rho)) &= -\beta G_{3,m}\left(\frac{1}{3} F_{1,m}(\alpha); \Delta\right); \quad \text{and} \\ M_2(\hat{\beta}^{PT}(\rho)) &= \frac{\sigma_v^2 K_1}{K_1 K_2 - K_3^2} \left\{ 1 - G_{3,m}\left(\frac{1}{3} F_{1,m}(\alpha); \Delta\right) \right. \\ &+ \left. \Delta\left(2G_{3,m}\left(\frac{1}{3} F_{1,m}(\alpha); \Delta\right) - G_{5,m}\left(\frac{1}{5} F_{1,m}(\alpha); \Delta\right)\right) \right\}. \end{aligned} \tag{3.2}$$

(iii) Shrinkage estimator (SE):

$$\begin{aligned} b_3(\hat{\beta}^S(\rho)) &= -\frac{cE(\tilde{\sigma}_v)K_1}{K_1K_2-K_3^2}\{1-2\Phi(-\Delta)\}; \quad E(\tilde{\sigma}_v)=\sigma_v K_n; \quad \text{and} \\ M_3(\hat{\beta}^S(\rho)) &= \frac{\sigma_v^2 K_1}{K_1K_2-K_3^2}\left\{1-\frac{2}{\pi}K_n^2\left[2e^{-\frac{\Delta}{2}}-1\right]\right\}, \end{aligned} \tag{3.3}$$

where $c=\sqrt{\frac{2}{\pi}}K_n$ and $K_n=\sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$.

3.2 Bias and MSE Expressions of θ

Following Saleh (2006), the bias and MSEs of $\tilde{\theta}(\rho)$, $\hat{\theta}(\rho)$, $\hat{\theta}^{PT}(\rho)$, and $\hat{\theta}^S(\rho)$ are presented below:

(i) Unrestricted estimator (UE):

$$\begin{aligned} b_1(\tilde{\theta}(\rho)) &= 0 \\ M_1(\tilde{\theta}(\rho)) &= \frac{\sigma_v^2 K_2}{K_1K_2-K_3^2}. \end{aligned} \tag{3.4}$$

(ii) Restricted estimator (RE):

$$\begin{aligned} b_2(\hat{\theta}(\rho)) &= \beta\frac{K_3}{K_1} \\ M_2(\hat{\theta}(\rho)) &= \frac{\sigma_v^2 K_2}{K_1K_2-K_3^2}-\frac{\sigma_v^2K_3^2}{K_1(K_1K_2-K_3^2)}\{1-\Delta\} \\ &= \frac{\sigma_v^2}{K_1}+\beta^2\left(\frac{K_3}{K_1}\right)^2. \end{aligned} \tag{3.5}$$

(iii) Preliminary test (PT) estimator:

$$\begin{aligned} b_3(\hat{\theta}^{PT}(\rho)) &= \beta\frac{K_3}{K_1}G_{3,m}\left(\frac{1}{3}F_{1,m}(\alpha);\Delta\right); \quad \text{and} \\ M_3(\hat{\theta}^{PT}(\rho)) &= \frac{\sigma_v^2K_2}{K_1K_2-K_3^2}-\frac{\sigma_v^2K_3^2}{K_1(K_1K_2-K_3^2)}\left\{G_{3,m}\left(\frac{1}{3}F_{1,m}(\alpha);\Delta\right)\right. \\ &- \left.\Delta\left(2G_{3,m}\left(\frac{1}{3}F_{1,m}(\alpha);\Delta\right)-G_{5,m}\left(\frac{1}{5}F_{1,m}(\alpha);\Delta\right)\right)\right\}. \end{aligned} \tag{3.6}$$

(iv) Shrinkage estimator (SE):

$$\begin{aligned} b_4(\hat{\theta}^S(\rho)) &= \frac{cE(\tilde{\sigma}_v)K_3}{K_1^{1/2}(K_1K_2-K_3^2)^{1/2}}\{1-2\Phi(-\Delta)\}; \quad E(\tilde{\sigma}_v)=\sigma_vK_n; \quad \text{and} \\ M_4(\hat{\theta}^S(\rho)) &= \sigma_v^2\left\{\frac{K_2}{K_1K_2-K_3^2}-\frac{2}{\pi}K_n\frac{K_3^2}{K_1(K_1K_2-K_3^2})\left(2e^{-\frac{\Delta}{2}}-1\right)\right\}, \end{aligned} \tag{3.7}$$

where $c=\sqrt{\frac{2}{\pi}}K_n$ and $K_n=\sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$.

4 Comparison

In this section we compare the performance of the proposed estimators in the sense of minimum mean squared error (MSE) criteria. The following two sections compare $\hat{\beta}^{PT}$ with $\tilde{\beta}$ and $\hat{\beta}^{S}$ with $\tilde{\beta}$ respectively.

4.1 Comparison for β

The mean-square relative efficiency (MRE) of the PT estimator compared to the unrestricted estimator $\tilde{\beta}$ is

$$\begin{aligned} MRE(\hat{\beta}^{PT}, \tilde{\beta}) &= \frac{MSE(\tilde{\beta})}{MSE(\hat{\beta}^{PT})} \\ &= \left[1 - G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)(1-2\Delta) - \Delta G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)\right]^{-1}. \end{aligned} \tag{4.1}$$

Under H_0 it has the maximum value

$$MRE(\hat{\beta}^{PT}, \hat{\beta}) = \left[1 - G_{3,m}\left(\frac{1}{3}F_{1,m}; 0\right)\right]^{-1} > 1. \tag{4.2}$$

Under the alternative hypothesis, $MRE(\hat{\beta}^{PT}, \tilde{\beta}) \geq 1$ according as

$$\Delta \leq \frac{G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)}{2G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right) - G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)} = \Delta_1(\alpha, \Delta) \text{ (say)}.$$

Hence $\hat{\beta}^{PT}$ performs better than $\tilde{\beta}$ when $\Delta \leq \Delta_1(\alpha, \Delta)$, otherwise $\tilde{\beta}$ is better than $\hat{\beta}^{PT}$. Since $2G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right) - G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right) \geq 0$, $\Delta_1(\alpha, \Delta) \leq 1$.

The mean-square relative efficiency(MRE) of S-estimator compared to the unrestricted estimator $\tilde{\beta}$ is

$$MRE(\hat{\beta}^{S}, \tilde{\beta}) = \frac{MSE(\tilde{\beta})}{MSE(\hat{\beta}^{S})} = \left[1 - \frac{2}{\pi}K_n\left[2e^{-\frac{\Delta}{2}} - 1\right]\right]^{-1}. \tag{4.3}$$

As $\Delta \to \infty$, then

$$MRE(\hat{\beta}^{S}, \tilde{\beta}) = \left[1 + \frac{2}{\pi}K_n\right]^{-1}. \tag{4.4}$$

Under H_0 it has the maximum value

$$MRE(\hat{\beta}^{S}, \tilde{\beta}) = \left[1 - \frac{2}{\pi}K_n\right]^{-1}. \tag{4.5}$$

In general, $MRE(\hat{\beta}^{S}, \tilde{\beta})$ decreases from $\left[1 - \frac{2}{\pi}K_n\right]^{-1}$ at $\Delta = 0$ and crosses the 1-line at $\Delta = 1.38$ and then drops to the minimum value $\left[1 + \frac{2}{\pi}K_n\right]^{-1}$ as $\Delta \to \infty$. The loss of efficiency is $1 - \left[1 + \frac{2}{\pi}K_n\right]^{-1}$ and the gain is $\left[1 - \frac{2}{\pi}K_n\right]^{-1}$. Thus for $0 \le \Delta \le 1.38$, θ_n^S performs better then $\tilde{\theta}$, otherwise $\tilde{\theta}$ performs better outside of this interval.

We have plotted the MRE vs Δ for fixed $\rho = 0.40$ and different values of n and α in Figure 1 and for fixed $n = 30$ and different values of α and ρ in Figure 2. It is clear from Figure 1 and 2 that the MRE of PT compared to UE is smaller for $\Delta > 1$ and that of SE relative to UE is smaller for $\Delta > 1.38$. Thus, the SE performs better than PT in the interval [0,1].

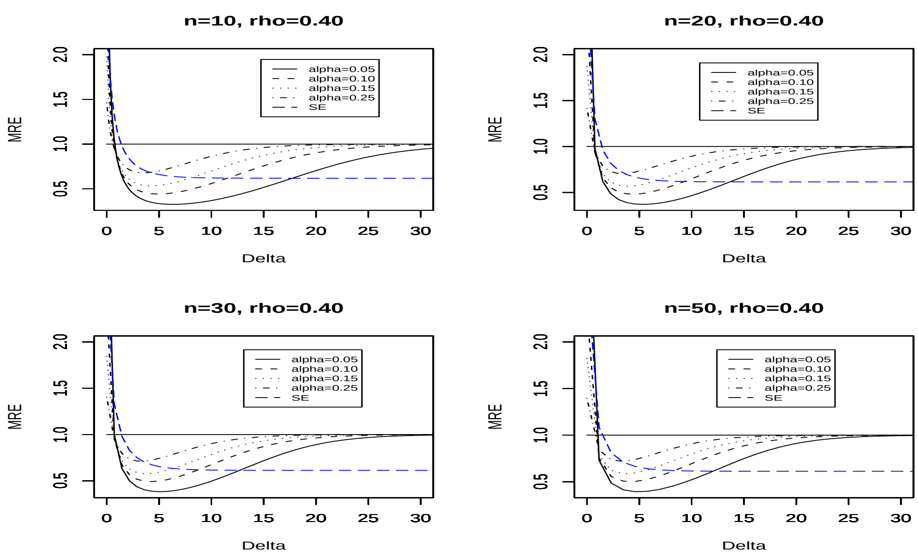

Figure 1: Graph of $MRE(\hat{\beta}^{PT}, \tilde{\beta})$ and $MRE(\hat{\beta}^{S}, \tilde{\beta})$ for different n and α

4.2 Comparison for θ

The MRE of the PT estimator compared to the unrestricted estimator $\tilde{\theta}$ is

$$\begin{aligned} MRE(\hat{\theta}^{PT}, \tilde{\theta}) &= \frac{MSE(\tilde{\theta})}{MSE(\hat{\theta}^{PT})} \\ &= \left[1 - \frac{K_3^2}{K_1K_2}\left\{G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)(1 - 2\Delta) + \Delta G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)\right\}\right]^{-1}. \end{aligned} \tag{4.6}$$

Under H_0 it has the maximum value.

$$MRE(\hat{\theta}^{PT}, \tilde{\theta}) = \frac{MSE(\tilde{\theta})}{MSE(\tilde{\theta}^{PT})} = \left[1 - \frac{K_3^2}{K_1K2}G_{3,m}\left(\frac{1}{3}F_{1,m}; 0\right)\right]^{-1}. \tag{4.7}$$

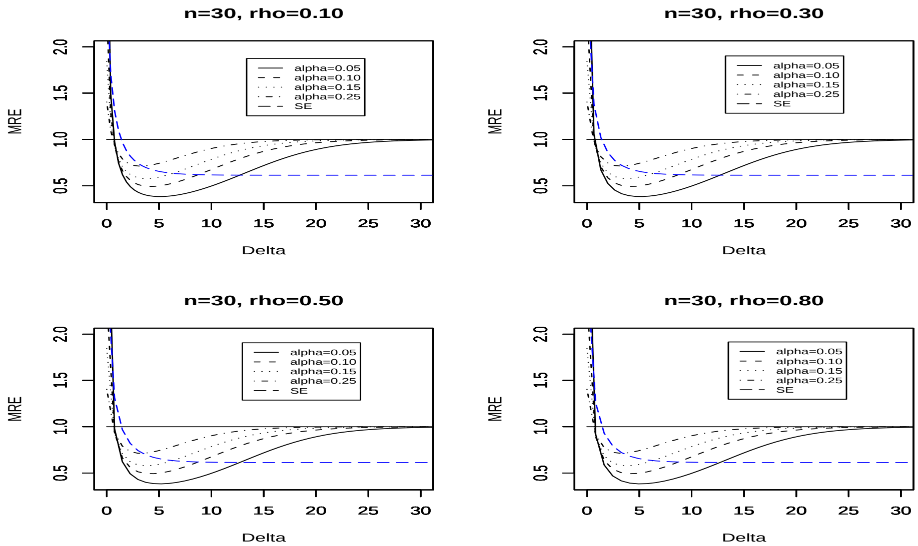

Figure 2: Graph of $MRE(\hat{\beta}^{PT}, \tilde{\beta})$ and $MRE(\hat{\beta}^{S}, \tilde{\beta})$ for different n and α

Under the alternative hypothesis, $MRE(\hat{\beta}^{PT}, \tilde{\beta}) \geq 1$ according as

$$\Delta \leq \frac{G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)}{2G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right) - G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)} = \Delta_1(\alpha, \Delta) \text{ (say).}$$

Hence $\hat{\theta}^{PT}$ performs better than $\tilde{\theta}$ when $\Delta \leq \Delta_1(\alpha, \Delta)$ otherwise $\tilde{\theta}$ is better than $\tilde{\theta}^{PT}$. Since $2G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right) - G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right) \geq 0$, $\Delta_1(\alpha, \Delta) \leq 1$.

The MRE of the SE compared to the unrestricted estimator $\tilde{\theta}$ is

$$MRE(\hat{\theta}^S, \tilde{\theta}) = \frac{MSE(\tilde{\theta})}{MSE(\hat{\theta}^S)} = \left[1 - \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\left[2e^{-\frac{\Delta}{2}} - 1\right]\right]^{-1}. \tag{4.8}$$

As $\Delta \to \infty$, we have $MRE(\hat{\theta}^S, \tilde{\theta}) = \left[1 + \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}$.

Under H_0 it has the maximum value

$$MRE(\hat{\theta}^S, \tilde{\beta}) = \left[1 - \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}. \tag{4.9}$$

In general, $MRE(\hat{\theta}^S, \tilde{\theta})$ decreases from $\left[1 - \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}$ at $\Delta = 0$ and cross the 1-line at $\Delta = 1.38$ and then drops to the minimum value $\left[1 + \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}$ as $\Delta \to \infty$. The loss of efficiency is $1 - \left[1 + \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}$ and the gain is $\left[1 - \frac{2}{\pi} K_n \frac{K_3^2}{K_1 K2}\right]^{-1}$. Thus for

$0 \leq \Delta \leq 1.38$, θ_n^S performs better then θ_n, otherwise θ_n performs better outside of this interval.

We have plotted MRE vs Δ for fixed $\rho = 0.40$ and different values of n and α in Figure 3 and for fixed $n = 30$ and different values of ρ and α in Figure 4. It is evident that the graphical analysis supports the comparison in section 4.2.

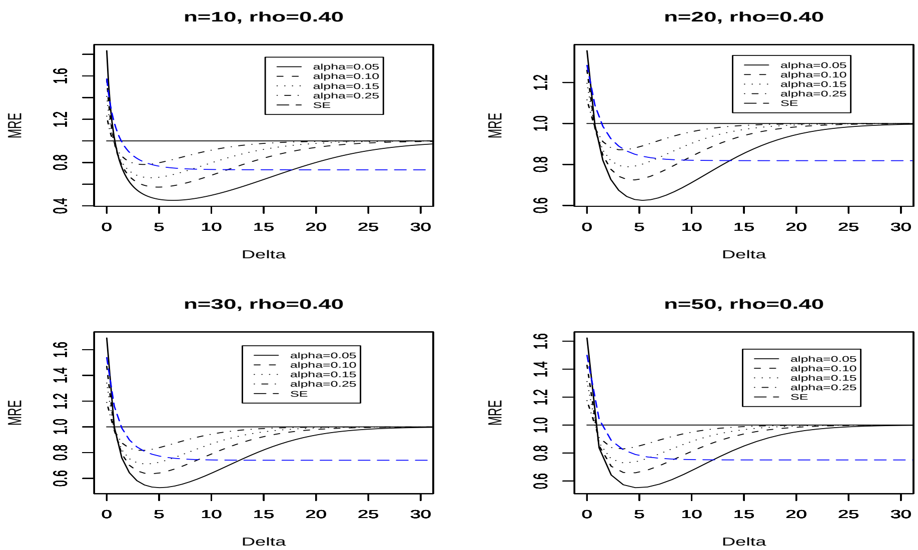

Figure 3: Graph of $MRE(\hat{\theta}^{PT}, \tilde{\theta})$ and $MRE(\hat{\theta}^{S}, \tilde{\theta})$ for $\rho = 0.4$ and different n and α

5 Relative Efficiency and Optimum Significance Level for PT

In this section, we describe the relative efficiency of the proposed estimators for β and θ. Accordingly, we provide max-min rule for the optimum choice of the level of significance for the PT estimator of β and θ.

5.1 Efficiency Analysis of $\hat{\beta}^{PT}$

Since the relative efficiency is a function of (α, Δ, ρ), we write

$$\begin{aligned} E(\alpha, \Delta, \rho) &= \frac{MSE(\tilde{\beta})}{MSE(\hat{\beta}^{PT})} \\ &= \left[1 - G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)(1 - 2\Delta) - \Delta G_{3,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)\right]^{-1}. \end{aligned} \tag{5.1}$$

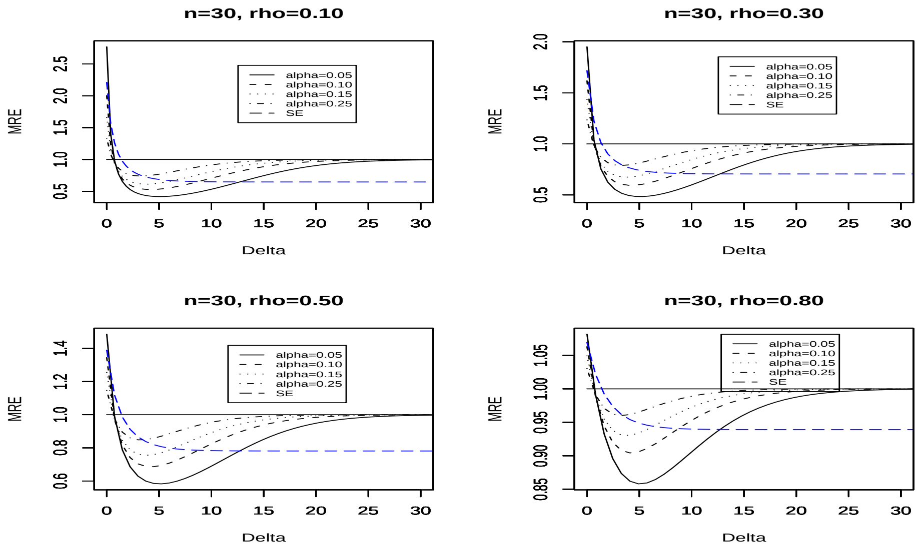

Figure 4: Graph of $MRE(\hat{\theta}^{PT}, \tilde{\theta})$ and $MRE(\hat{\theta}^{S}, \tilde{\theta})$ for n=20 and different values of ρ and α

For a given n and ρ, $E(\alpha, \Delta, \rho)$, is a function of α and Δ. For $\alpha \neq 0$, it has maximum at $\Delta = 0$ with value

$$E(\alpha, 0, 0) = \left[1 - G_{3,m}\left(\frac{1}{3}F_{1,m}; 0\right)\right]^{-1} > 1. \tag{5.2}$$

As Δ increases from 0, $E(\alpha, \Delta, \rho)$ decreases and crossing the line $E(\alpha, \Delta, \rho) = 1$ to a minimum $E(\alpha, \Delta^0, \rho)$ at $\Delta = \Delta^0$, then increases towards 1 as $\Delta \to \infty$. The value $E(\alpha, 0, 0)$ decreases as α increases. On the other hand, for $\alpha \neq 0$, as Δ varies the graphs of $E(0, \Delta, \rho)$ and $E(1, \Delta, \rho)$ intersect in the range $0 \leq \Delta \leq 1$. Usually, the values of Δ are unknown, then one might follow the *minimum guaranteed efficiency* procedures due to Han and Bancraft (1968) which in turn determine the optimum level of significance for given minimum guaranteed efficiency say E_{Min}. In this procedure one looks for a suitable α from the set $A_\alpha = \{\alpha | E(\alpha, \Delta, \rho) \geq E_{Min}\}$. The PT is chosen for which $E(\alpha, \Delta, \rho)$ is maximized over all $\alpha \in A_\alpha$ and Δ. Thus one solves the equation

$$\min_{\Delta} E(\alpha, \Delta_{Min}(\alpha), \rho) = E_{Min}. \tag{5.3}$$

The solution α^* for (5.3) is the optimum level of significance for the PT with minimum guaranteed efficiency E_{Min} which may increase to $E(\alpha^*, 0, \rho)$. The minimum and maximum guaranteed efficiency of the $\hat{\beta}^{PT}$ for $\alpha = 0.05(0.05), 0.40, 0.50$, $\rho = 0(0.1)0.90$ and for $n = 30$ and $n = 50$ are provided in Table 1 and Table 2 respectively. The first two rows of Table

1 & 2 give the maximum and minimum guaranteed efficiency for PT estimator. The third row gives the value of $\Delta = \Delta_{min}$ for which the minimum efficiency of PT (in second row) and the efficiency of SE (in fourth row) are obtained. From these tables we observe that the relative efficiencies are function of Δ, α and ρ. We also observed that the SE performs better than the PT for smaller α and the PT performs better than the SE for larger values of α. For more on optimal significance level we refer to Kibria and Saleh (2005, 2006) among others.

5.2 Efficiency Analysis of $\hat{\theta}^{PT}$

The relative efficiency of $\hat{\theta}^{PT}$ is a function of α, Δ and ρ and defined as

$$
\begin{aligned}
E(\alpha, \Delta, \rho) &= \frac{MSE(\tilde{\theta})}{MSE(\hat{\theta}^{PT})} \\
&= \left[1 - \frac{K_3^2}{K_1 K2}\left\{G_{3,m}\left(\frac{1}{3}F_{1,m}; \Delta\right)(1-2\Delta) + \Delta G_{5,m}\left(\frac{1}{5}F_{1,m}; \Delta\right)\right\}\right]^{-1} \qquad (5.4)
\end{aligned}
$$

For a given n and ρ, $E(\alpha, \Delta, \rho)$, is a function of α and Δ. For $\alpha \neq 0$, it has maximum at $\Delta = 0$ with value

$$
E(\alpha, 0, \rho) = \left[1 - \frac{K_3^2}{K_1 K2} G_{3,m}\left(\frac{1}{3}F_{1,m}; 0\right)\right]^{-1}
$$

Now following the same procedures as in section 5.1, we have provided the minimum and maximum guaranteed efficiency of the $\hat{\beta}^{PT}$ for $\alpha = 0.05(0.05), 0.40, 0.50$, $\rho = 0(0.1)0.90$ and for $n = 30$ and $n = 50$ are provided in Table 3 and Table 4 respectively. The first two rows of Table 3 & 4 give the maximum and minimum guaranteed efficiency of shrinkage estimator (SE). The remaining rows of these tables, contain the maximum and minimum guaranteed efficiency for PT estimator, the value of $\Delta = \Delta_{min}$ for which the minimum guaranteed efficiency of PT (EminPT) and the efficiency of SE ($\mathrm{ES}_{\Delta_{min}}$) are obtained. From these tables we observe that both relative efficiencies are function of Δ, α and ρ. We also observed that the SE performs better than the PT for smaller α and the PT performs better than the SE for larger values of α. We have plotted ρ vs maximum guaranteed efficiency in Figure 5 (left panel) and ρ vs minimum guaranteed efficiency in Figure 5 (right panel). From these graphs we can see that the maximum guaranteed efficiency is decreasing function of ρ while the minimum guaranteed efficiency is increasing function of ρ. We also observe that for any ρ, the SE dominates the PTE for small level of significance while the PTE dominates the SE for large level of significance.

6 Concluding Remarks

This paper deals with the problem of estimation of the intercept and slope parameters of a simple linear model with autocorrelated errors where ρ is known. Accordingly, the unrestricted estimator, restricted estimator, preliminary test estimator and shrinkage estimators

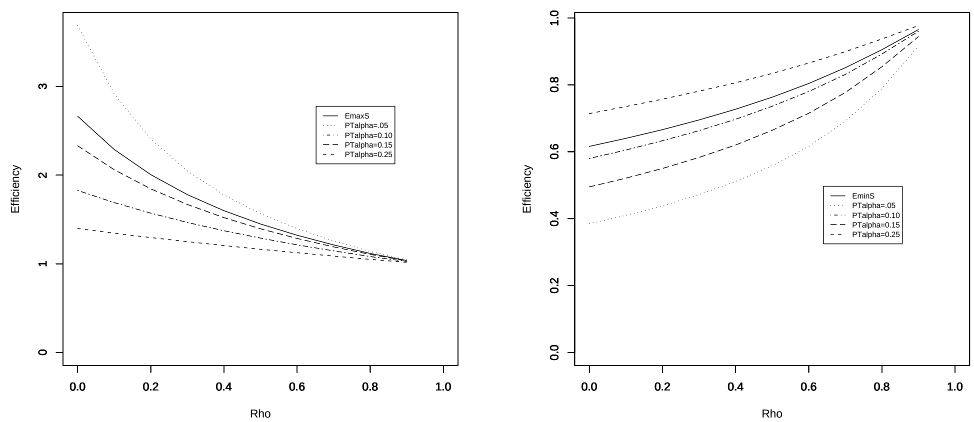

Figure 5: Graph of maximum guaranteed efficiency vs ρ for $n = 30$ (left panel) and graph of minimum guaranteed efficiency vs ρ for $n = 30$ (right panel)

of the intercept θ and slope β are defined. The bias and MSE expressions of the proposed estimators are obtained and compared. For some selected parameters, graphs and the relative efficiency tables are presented. It is shown that the proposed preliminary test and shrinkage estimators dominate the unrestricted estimator for some values of Δ. Some tables of maximum/minimum efficiencies are provided for various ρ and α to assess the performance of the estimators and choose optimum levels of significance of the PTE. The minimum guaranteed efficiencies are increasing function of ρ while the maximum guaranteed efficiencies are decreasing function of ρ for both PTE and SEs. We also observe that none of the PTE and SE dominate each other uniformly. However, considering the overall performance of SE relative to PTE, SE is preferable to PTE, because it produces interpolated estimates that are free of the size of the test. However, if one is willing to accept the larger size of the test (say 0.25 or higher), he or she might consider PT estimator for estimating the parameters.

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Table 1: Maximum and Minimum Guaranteed Efficiens of $\hat{\beta}^{PT}$ for $n = 30$

α	ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	EmaxS	2.710	2.710	2.710	2.710	2.710	2.710	2.710	2.710	2.710	2.710
	EminS	0.613	0.613	0.613	0.613	0.613	0.613	0.613	0.613	0.613	0.613
0.05	EmaxPT	3.790	3.790	3.790	3.790	3.790	3.790	3.790	3.790	3.790	3.790
	EminPT	0.383	0.383	0.383	0.383	0.384	0.385	0.385	0.386	0.384	0.384
	$\text{ES}_{\Delta_{min}}$	0.653	0.656	0.651	0.656	0.647	0.644	0.643	0.641	0.661	0.662
	Δ_{min}	5.080	4.933	5.166	4.952	5.401	5.591	5.640	5.728	4.724	4.705
0.10	EmaxPT	2.363	2.363	2.363	2.363	2.363	2.363	2.363	2.363	2.363	2.363
	EminPT	0.492	0.492	0.492	0.495	0.493	0.492	0.492	0.492	0.495	0.495
	$\text{ES}_{\Delta_{min}}$	0.676	0.676	0.680	0.697	0.683	0.678	0.676	0.674	0.661	0.662
	Δ_{min}	4.234	4.229	4.133	3.714	4.051	4.193	4.230	4.296	4.724	4.705
0.15	EmaxPT	1.843	1.843	1.843	1.843	1.843	1.843	1.843	1.843	1.843	1.843
	EminPT	0.578	0.578	0.579	0.578	0.579	0.580	0.580	0.581	0.583	0.587
	$\text{ES}_{\Delta_{min}}$	0.693	0.707	0.680	0.697	0.683	0.678	0.676	0.674	0.730	0.662
	Δ_{min}	3.810	3.524	4.133	3.714	4.051	4.193	4.230	4.296	3.149	4.705
0.20	EmaxPT	1.571	1.571	1.571	1.571	1.571	1.571	1.571	1.571	1.571	1.571
	EminPT	0.650	0.650	0.652	0.650	0.654	0.656	0.657	0.656	0.651	0.667
	$\text{ES}_{\Delta_{min}}$	0.715	0.707	0.734	0.697	0.683	0.678	0.756	0.752	0.730	0.662
	Δ_{min}	3.387	3.524	3.100	3.714	4.051	4.193	2.820	2.864	3.149	4.705
0.25	EmaxPT	1.404	1.404	1.404	1.404	1.404	1.404	1.404	1.404	1.404	1.404
	EminPT	0.712	0.713	0.712	0.715	0.717	0.716	0.715	0.715	0.712	0.727
	$\text{ES}_{\Delta_{min}}$	0.728	0.707	0.734	0.697	0.767	0.758	0.756	0.752	0.730	0.805
	Δ_{min}	3.175	3.524	3.100	3.714	2.701	2.796	2.820	2.864	3.149	2.353
0.30	EmaxPT	1.292	1.292	1.292	1.292	1.292	1.292	1.292	1.292	1.292	1.292
	EminPT	0.766	0.767	0.766	0.771	0.769	0.768	0.767	0.767	0.766	0.776
	$\text{ES}_{\Delta_{min}}$	0.728	0.756	0.734	0.697	0.767	0.758	0.756	0.752	0.730	0.805
	Δ_{min}	3.175	2.819	3.100	3.714	2.701	2.796	2.820	2.864	3.149	2.353
0.35	EmaxPT	1.213	1.213	1.213	1.213	1.213	1.213	1.213	1.213	1.213	1.213
	EminPT	0.813	0.814	0.813	0.817	0.814	0.814	0.814	0.813	0.814	0.819
	$\text{ES}_{\Delta_{min}}$	0.744	0.756	0.734	0.791	0.767	0.758	0.756	0.752	0.730	0.805
	Δ_{min}	2.964	2.819	3.100	2.476	2.701	2.796	2.820	2.864	3.149	2.353
0.40	EmaxPT	1.156	1.156	1.156	1.156	1.156	1.156	1.156	1.156	1.156	1.156
	EminPT	0.854	0.854	0.854	0.856	0.854	0.854	0.854	0.854	0.854	0.857
	$\text{ES}_{\Delta_{min}}$	0.744	0.756	0.734	0.791	0.767	0.758	0.756	0.752	0.730	0.805
	Δ_{min}	2.964	2.819	3.100	2.476	2.701	2.796	2.820	2.864	3.149	2.353
0.50	EmaxPT	1.081	1.081	1.081	1.081	1.081	1.081	1.081	1.081	1.081	1.081
	EminPT	0.916	0.916	0.917	0.917	0.916	0.916	0.916	0.916	0.917	0.917
	$\text{ES}_{\Delta_{min}}$	0.762	0.756	0.734	0.791	0.767	0.758	0.756	0.752	0.730	0.805
	Δ_{min}	2.752	2.819	3.100	2.476	2.701	2.796	2.820	2.864	3.149	2.353

Table 2: Maximum and Minimum Guaranteed Efficiens of $\hat{\beta}^{PT}$ for $n = 50$

α	ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	EmaxS	2.727	2.727	2.727	2.727	2.727	2.727	2.727	2.727	2.727	2.727
	EminS	0.612	0.612	0.612	0.612	0.612	0.612	0.612	0.612	0.612	0.612
0.05	EmaxPT	3.702	3.702	3.702	3.702	3.702	3.702	3.702	3.702	3.702	3.702
	EminPT	0.392	0.396	0.396	0.392	0.394	0.395	0.394	0.394	0.395	0.405
	$\text{ES}_{\Delta_{min}}$	0.659	0.673	0.676	0.653	0.645	0.644	0.645	0.648	0.645	0.694
	Δ_{min}	4.788	4.305	4.223	5.033	5.431	5.527	5.432	5.300	5.480	3.780
0.10	EmaxPT	2.327	2.327	2.327	2.327	2.327	2.327	2.327	2.327	2.327	2.327
	EminPT	0.501	0.501	0.501	0.509	0.517	0.519	0.517	0.514	0.518	0.502
	$\text{ES}_{\Delta_{min}}$	0.684	0.673	0.676	0.653	0.645	0.644	0.645	0.648	0.645	0.694
	Δ_{min}	3.990	4.305	4.223	5.033	5.431	5.527	5.432	5.300	5.480	3.780
0.15	EmaxPT	1.822	1.822	1.822	1.822	1.822	1.822	1.822	1.822	1.822	1.822
	EminPT	0.585	0.589	0.588	0.604	0.601	0.599	0.601	0.604	0.600	0.585
	$\text{ES}_{\Delta_{min}}$	0.703	0.673	0.676	0.653	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	3.591	4.305	4.223	5.033	2.716	2.764	2.716	2.650	2.740	3.780
0.20	EmaxPT	1.557	1.557	1.557	1.557	1.557	1.557	1.557	1.557	1.557	1.557
	EminPT	0.656	0.661	0.664	0.671	0.664	0.663	0.664	0.666	0.664	0.658
	$\text{ES}_{\Delta_{min}}$	0.703	0.751	0.676	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	3.591	2.870	4.223	2.516	2.716	2.764	2.716	2.650	2.740	3.780
0.25	EmaxPT	1.395	1.395	1.395	1.395	1.395	1.395	1.395	1.395	1.395	1.395
	EminPT	0.717	0.719	0.729	0.726	0.722	0.721	0.722	0.723	0.721	0.721
	$\text{ES}_{\Delta_{min}}$	0.726	0.751	0.676	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	3.192	2.870	4.223	2.516	2.716	2.764	2.716	2.650	2.740	3.780
0.30	EmaxPT	1.286	1.286	1.286	1.286	1.286	1.286	1.286	1.286	1.286	1.286
	EminPT	0.771	0.771	0.785	0.776	0.773	0.772	0.773	0.774	0.772	0.777
	$\text{ES}_{\Delta_{min}}$	0.726	0.751	0.676	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	3.192	2.870	4.223	2.516	2.716	2.764	2.716	2.650	2.740	3.780
0.35	EmaxPT	1.209	1.209	1.209	1.209	1.209	1.209	1.209	1.209	1.209	1.209
	EminPT	0.817	0.817	0.829	0.820	0.818	0.817	0.818	0.818	0.818	0.824
	$\text{ES}_{\Delta_{min}}$	0.758	0.751	0.838	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	2.793	2.870	2.112	2.516	2.716	2.764	2.716	2.650	2.740	3.780
0.40	EmaxPT	1.152	1.152	1.152	1.152	1.152	1.152	1.152	1.152	1.152	1.152
	EminPT	0.857	0.856	0.865	0.858	0.857	0.857	0.857	0.857	0.857	0.864
	$\text{ES}_{\Delta_{min}}$	0.758	0.751	0.838	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	2.793	2.870	2.112	2.516	2.716	2.764	2.716	2.650	2.740	3.780
0.50	EmaxPT	1.079	1.079	1.079	1.079	1.079	1.079	1.079	1.079	1.079	1.079
	EminPT	0.918	0.918	0.921	0.918	0.918	0.918	0.918	0.918	0.918	0.924
	$\text{ES}_{\Delta_{min}}$	0.758	0.751	0.838	0.785	0.765	0.760	0.765	0.771	0.762	0.694
	Δ_{min}	2.793	2.870	2.112	2.516	2.716	2.764	2.716	2.650	2.740	3.780

Table 3: Maximum and Minimum Guaranteed Efficiens of $\hat{\theta}^{PT}$ for $n = 30$

α	ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	EmaxS	2.556	2.200	1.926	1.707	1.528	1.378	1.251	1.145	1.062	1.013
	EminS	0.622	0.647	0.675	0.707	0.743	0.785	0.833	0.888	0.945	0.987
0.05	EmaxPT	3.450	2.750	2.277	1.935	1.675	1.470	1.305	1.173	1.073	1.015
	EminPT	0.391	0.418	0.449	0.487	0.531	0.589	0.663	0.758	0.870	0.968
	$ES_{\Delta_{min}}$	0.662	0.680	0.716	0.749	0.773	0.808	0.851	0.899	0.953	0.989
	Δ_{min}	5.032	5.356	4.897	4.710	5.173	5.415	5.555	5.791	4.960	5.175
0.10	EmaxPT	2.255	1.995	1.784	1.609	1.461	1.334	1.224	1.131	1.057	1.012
	EminPT	0.501	0.529	0.561	0.599	0.640	0.691	0.753	0.829	0.914	0.980
	$ES_{\Delta_{min}}$	0.685	0.715	0.743	0.749	0.802	0.832	0.869	0.911	0.953	0.989
	Δ_{min}	4.193	4.017	3.918	4.710	3.880	4.061	4.167	4.343	4.960	5.175
0.15	EmaxPT	1.790	1.654	1.535	1.429	1.334	1.248	1.170	1.101	1.044	1.009
	EminPT	0.586	0.613	0.643	0.676	0.714	0.760	0.813	0.874	0.937	0.987
	$ES_{\Delta_{min}}$	0.702	0.715	0.743	0.786	0.802	0.832	0.869	0.911	0.965	0.989
	Δ_{min}	3.774	4.017	3.918	3.532	3.880	4.061	4.167	4.343	3.307	5.175
0.20	EmaxPT	1.540	1.458	1.383	1.313	1.248	1.187	1.131	1.078	1.035	1.007
	EminPT	0.658	0.682	0.711	0.739	0.774	0.813	0.857	0.905	0.952	0.990
	$ES_{\Delta_{min}}$	0.712	0.746	0.743	0.786	0.802	0.832	0.869	0.937	0.965	0.994
	Δ_{min}	3.564	3.348	3.918	3.532	3.880	4.061	4.167	2.895	3.307	2.587
0.25	EmaxPT	1.384	1.331	1.281	1.233	1.187	1.143	1.101	1.061	1.027	1.006
	EminPT	0.719	0.741	0.766	0.791	0.822	0.854	0.888	0.926	0.964	0.992
	$ES_{\Delta_{min}}$	0.724	0.746	0.794	0.786	0.802	0.883	0.909	0.937	0.965	0.994
	Δ_{min}	3.354	3.348	2.938	3.532	3.880	2.707	2.778	2.895	3.307	2.587
0.30	EmaxPT	1.279	1.243	1.208	1.174	1.141	1.109	1.077	1.047	1.021	1.005
	EminPT	0.773	0.792	0.812	0.835	0.860	0.884	0.912	0.943	0.973	0.994
	$ES_{\Delta_{min}}$	0.737	0.746	0.794	0.786	0.865	0.883	0.909	0.937	0.965	0.994
	Δ_{min}	3.145	3.348	2.938	3.532	2.587	2.707	2.778	2.895	3.307	2.587
0.35	EmaxPT	1.204	1.179	1.155	1.130	1.106	1.083	1.059	1.036	1.017	1.004
	EminPT	0.819	0.836	0.851	0.872	0.890	0.910	0.932	0.956	0.979	0.995
	$ES_{\Delta_{min}}$	0.753	0.794	0.794	0.786	0.865	0.883	0.909	0.937	0.965	0.994
	Δ_{min}	2.935	2.678	2.938	3.532	2.587	2.707	2.778	2.895	3.307	2.587
0.40	EmaxPT	1.149	1.132	1.114	1.097	1.080	1.062	1.045	1.028	1.013	1.003
	EminPT	0.858	0.871	0.884	0.902	0.915	0.931	0.948	0.967	0.985	0.997
	$ES_{\Delta_{min}}$	0.753	0.794	0.794	0.863	0.865	0.883	0.909	0.937	0.965	0.994
	Δ_{min}	2.935	2.678	2.938	2.355	2.587	2.707	2.778	2.895	3.307	2.587
0.50	EmaxPT	1.078	1.069	1.060	1.052	1.043	1.034	1.024	1.015	1.007	1.002
	EminPT	0.919	0.927	0.935	0.944	0.952	0.962	0.972	0.982	0.992	0.998
	$ES_{\Delta_{min}}$	0.771	0.794	0.794	0.863	0.865	0.883	0.909	0.937	0.965	0.994
	Δ_{min}	2.725	2.678	2.938	2.355	2.587	2.707	2.778	2.895	3.307	2.587

Table 4: Maximum and Minimum Guaranteed Efficiens of $\hat{\theta}^{PT}$ for $n = 50$

α	ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	EmaxS	2.545	2.182	1.904	1.683	1.504	1.355	1.230	1.128	1.051	1.009
	EminS	0.622	0.649	0.678	0.711	0.749	0.792	0.842	0.898	0.953	0.991
0.05	EmaxPT	3.330	2.661	2.208	1.879	1.629	1.433	1.275	1.150	1.060	1.011
	EminPT	0.403	0.432	0.468	0.503	0.550	0.610	0.688	0.786	0.901	0.978
	$ES_{\Delta_{min}}$	0.662	0.682	0.704	0.755	0.781	0.817	0.860	0.909	0.957	0.992
	Δ_{min}	5.042	5.328	5.757	4.586	5.026	5.266	5.429	5.720	6.666	5.379
0.10	EmaxPT	2.205	1.952	1.746	1.576	1.432	1.309	1.203	1.114	1.046	1.008
	EminPT	0.511	0.540	0.573	0.612	0.662	0.718	0.783	0.860	0.931	0.987
	$ES_{\Delta_{min}}$	0.688	0.717	0.749	0.755	0.781	0.817	0.860	0.909	0.970	0.992
	Δ_{min}	4.125	3.996	3.838	4.586	5.026	5.266	5.429	5.720	3.333	5.379
0.15	EmaxPT	1.762	1.628	1.511	1.407	1.314	1.230	1.154	1.088	1.036	1.007
	EminPT	0.595	0.623	0.653	0.695	0.742	0.787	0.836	0.892	0.948	0.991
	$ES_{\Delta_{min}}$	0.708	0.717	0.749	0.755	0.781	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	3.667	3.996	3.838	4.586	5.026	2.633	2.714	2.860	3.333	5.379
0.20	EmaxPT	1.522	1.441	1.367	1.298	1.234	1.174	1.118	1.068	1.028	1.005
	EminPT	0.666	0.694	0.720	0.761	0.794	0.829	0.870	0.916	0.961	0.994
	$ES_{\Delta_{min}}$	0.734	0.717	0.749	0.755	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	3.209	3.996	3.838	4.586	2.513	2.633	2.714	2.860	3.333	5.379
0.25	EmaxPT	1.372	1.319	1.269	1.222	1.176	1.133	1.091	1.053	1.022	1.004
	EminPT	0.726	0.753	0.776	0.811	0.834	0.863	0.898	0.935	0.971	0.995
	$ES_{\Delta_{min}}$	0.734	0.796	0.749	0.871	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	3.209	2.664	3.838	2.293	2.513	2.633	2.714	2.860	3.333	5.379
0.30	EmaxPT	1.271	1.234	1.200	1.166	1.133	1.101	1.070	1.041	1.017	1.003
	EminPT	0.778	0.800	0.823	0.848	0.867	0.892	0.920	0.950	0.978	0.997
	$ES_{\Delta_{min}}$	0.734	0.796	0.749	0.871	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	3.209	2.664	3.838	2.293	2.513	2.633	2.714	2.860	3.333	5.379
0.35	EmaxPT	1.198	1.173	1.149	1.124	1.100	1.077	1.054	1.032	1.014	1.002
	EminPT	0.824	0.840	0.863	0.879	0.896	0.916	0.938	0.961	0.983	0.998
	$ES_{\Delta_{min}}$	0.769	0.796	0.749	0.871	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	2.750	2.664	3.838	2.293	2.513	2.633	2.714	2.860	3.333	5.379
0.40	EmaxPT	1.145	1.128	1.110	1.093	1.075	1.058	1.041	1.024	1.010	1.002
	EminPT	0.862	0.875	0.895	0.906	0.920	0.935	0.953	0.971	0.987	0.998
	$ES_{\Delta_{min}}$	0.769	0.796	0.749	0.871	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	2.750	2.664	3.838	2.293	2.513	2.633	2.714	2.860	3.333	5.379
0.50	EmaxPT	1.076	1.067	1.058	1.049	1.040	1.031	1.022	1.013	1.006	1.001
	EminPT	0.921	0.929	0.942	0.947	0.955	0.964	0.974	0.984	0.993	0.999
	$ES_{\Delta_{min}}$	0.769	0.796	0.900	0.871	0.874	0.892	0.917	0.944	0.970	0.992
	Δ_{min}	2.750	2.664	1.919	2.293	2.513	2.633	2.714	2.860	3.333	5.379