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SOME IMPUTATION METHODS FOR NON-RESPONSE AT CURRENT OCCASION IN TWO-OCCASION ROTATION PATTERNS

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summary

Among modern strategies applied to cope with the problems of non-response, in sample surveys, imputation is one of them. Imputation is the filling up method of incomplete data by adapting the standard analytic model in statistics. The purpose of the present work is to study the intelligible use of imputation methods in dealing with non-response at current occasion in two-occasion successive (rotation) sampling. Chain-type regressions in ratio estimators have been proposed for estimating the population mean at current occasion. Expressions for optimum estimators and their mean square errors have been derived. To study the effectiveness of the suggested imputation methods, performances of the proposed estimators are compared in two different situations: with and without non-response. Behaviors of the proposed estimators are demonstrated through empirical studies.

Keywords and phrases: Missing data, Successive sampling, Chain-type regression in ratio, Optimum replacement policy.

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1 Introduction

There are many situations in agricultural, demographic and social surveys, where the sampling units have to be observed a number of times at specified time intervals to estimate the change in population parameters or to know their current estimates. Surveys, where units spread over space and observations spread over time are defined in literatures as longitudinal surveys. In such surveys practitioners are often concerned with measuring characteristics of a population on several occasions to estimate the population means (or totals) of the characteristics or to study a pattern of variation in these parameters over the different occasions. For example, in an agricultural survey one may be interested in (*i*) estimating the average amount of yield per acre of an important crop (say wheat) in current season, (*ii*) estimating the change in average amount of yield for a province (county) for two different seasons, and (*iii*) estimating both parameters from (*i*) and (*ii*) simultaneously. Successive (rotation) sampling provides a strong statistical tool for generating the reliable estimates of population characteristics at different occasions. Theory of rotation (successive) sampling appears to have started with the work of Jessen (1942), in which entire information collected in the previous investigations (occasions) was used for estimation at the current occasion. This theory was extended by Patterson (1950), Rao and Graham (1964), Gupta (1979) and Das (1982), among others. Sen (1971) developed estimators for the population mean on the current occasion using information on two auxiliary variates available on previous occasion. Sen (1972, 73) extended his work for more than two auxiliary variates. In addition to the information from previous occasion, Singh et al. (1991) and Singh and Singh (2001) used information on an auxiliary variate available only on the current occasion for estimating the current population mean in two-occasion successive sampling. Singh (2003) generalized his estimation procedures for h-occasions successive sampling. In many situations, information on an auxiliary variate may be readily available on the first as well as on the second occasion, for example (*i*) total cultivated area in agricultural survey is known, (*ii*) number of academic institutions with their intake capacity is well known in an educational survey and (*iii*) number of polluting industries is known in sample surveys on environment. Utilizing the auxiliary information available on both the occasions, Feng and Zou (1997), Biradar and Singh (2001), Singh (2005), Singh and Priyanka (2006, 2007 and 2008) have proposed several chain-type ratio, difference and regression estimators for estimating the population mean at the current (second) occasion in two-occasion successive sampling.

In sample survey, non-response is one of the major problems encountered by survey statisticians. Longitudinal surveys are more prone to this problem than single-occasion surveys. For example, in agricultural surveys, it might be possible that crop on certain plots is destroyed due to some natural calamities or disease so that yield on these plots is impossible to be measured. Such non-response (incompleteness) can have different patterns and causes. It is well recognized by survey statisticians that, if the suitable information about the nature of non-response in the population is unknown, the inference concerning population parameters could be spoiled. Imputation is one of the many methods used to minimize the negative effect of non-response in survey data. Imputation deals with the filling of missing item values in a data by artificial value. To deal with missing values effectively, Sande (1979) and Kalton et al. (1981) suggested imputation methods that make incomplete data sets structurally complete and its analysis simple. Kalton et al. (1982) and Singh and Singh (1991) suggested useful imputation methods for surveys in which one uses an estimation procedure based on complete data set and discards data for all those units for which information is not available for at least one time stage. Imputation may also be carried out with the aid of an auxiliary variate, if such is available. For example, Lee et al. (1994, 1995) used the information on an available auxiliary variate for imputation purpose. Later Singh and Horn (2002) suggested a compromised method of imputation. Further utilizing auxiliary information Ahmed et al. (2006) and Singh (2009) suggested several new imputation based methods to reduce the effect of non-response in sample surveys.

Motivated with the above arguments Singh et al. (2008) discussed some reliable imputation methods for the estimation of population mean at the current occasion in two-occasion successive (rotation) sampling. Following the work of Singh et al. (2008), the objective of the present work is to study the effect of non-response at current occasion in two-occasion successive (rotation) sampling. It is assumed that all units in the sample respond at the first occasion. Since, the units are responding at the first occasion, therefore they are familiar with the situations, hence they are expected to co-operate and will respond at the current (second) occasion in the matched portions of the units. For example, in economic surveys, the respondents who have co-operated at the first occasion and now they are well acquainted with the pros and cons of the situations, they are supposed to co-operate at the second occasion as well. At the current occasion a sample is drawn afresh from the remaining units of the population, which has not been sampled at the previous occasion, so there is a possibility of non-response in fresh sample at the current (second) occasion because they are not familiar with the pros and cons of the situations. In the light of above discussions, chain-type regression in ratio estimators, that use imputation are proposed for estimating the population mean at the current (second) occasion in two occasions successive (rotation) sampling. The performance of the proposed estimators is compared between two different situations: with and without non-response and subsequent recommendation regarding the choice of an appropriate imputation based estimation technique is made.

2 Notation and Proposed Estimators

Consider a finite population $U = (U_1, U_2, \ldots, U_N)$ of N units has been sampled over two occasions. The character under study is denoted by $x(y)$ on the first (second) occasion, respectively. Let information on an auxiliary variable z, with the known population mean, be available on both occasions. A simple random sample (without replacement) s_n of n units is drawn on the first occasion and it is assumed that we get complete response from these units. A random sub-sample s_m of $m = n\lambda$ units is retained (matched) from s_n for its use on the current (second) occasion and it is further assumed these matched units are completely responding at current occasion as well. A fresh simple random sample (without replacement) s_u of $u = (n - m) = n\lambda$ units is drawn on the current (second) occasion from the non-sampled units of the population so that the sample size on the current occasion remains n. We assume that non-response occurs in the fresh sample drawn at the current occasion. Let the number of responding units out of sampled u units, which are drawn afresh at current occasion, be denoted by r, the set of responding units in s_u by R_u, and that of non-responding units by R_u^c . λ and μ $(\lambda + \mu = 1)$ are the fractions of the matched and fresh sample, respectively, at current occasion. For every unit $i \in R_u$ the value y_i is observed, but for the units $i \in R_u^c$ the y_i values are missing and instead imputed values are derived. The following notations have been considered in this work:

$\bar{X}, \bar{Y}, \bar{Z}$: Population mean of x, y and z respectively

$\bar{x}_n, \bar{x}_m, \bar{y}_m, \bar{z}_u, \bar{z}_n, \bar{z}_m$: Sample means of the respective variables of the of the sample sizes shown in suffices

$\bar{y}_r, \bar{z}_r$: Response means of y and z respectively

$\rho_{yx}, \rho_{yz}, \rho_{xz}$: Correlation coefficient between the variables shown in suffices

$S_x^2 = (N-1)^{-1} \sum_{i=1}^{N} (x_i - \bar{X})^2$: Population mean square of x

S_y^2, S_z^2 : Population mean square of y, z respectively

$f_1 (= \frac{r}{u})$: The fraction of respondents in the sample of size u

$t = (1 - f_1)$: The fraction of non-respondents in the sample of size u

To estimate the population mean $\bar{Y}$ on the current (second) occasion, two different sets of estimators are considered. One set of estimators $T_u = \{T_{1u}, T_{2u}, T_{3u}\}$ based on sample s_u of size $u(= n\mu)$ drawn afresh on the second occasion and the second set of estimators $T_m = \{T_{1m}, T_{2m}\}$ based on the sample s_m of size $m(= n\lambda$) common with both the occasions. Estimators T_{1u}, T_{2u} and T_{3u} of the set T_u are structured to cope up with the problems of non-response at the current occasion. The missing values are replaced by calibrate imputed values using the ratio and regression methods of imputation. Following three different imputation techniques for imputing the missing values at current occasion have been considered:

$$(i) \qquad y_{.i} = \begin{cases} y_i & \text{if } i \in R_u \\ \bar{y}_r + \hat{b}_1 \left[\frac{u(\bar{Z} - \bar{z}_r)}{u-r} + z_i - \bar{z}_r \right] & \text{if } i \in R_u^c, \end{cases} \tag{2.1}$$

where

$$\hat{b}_1 = \frac{s_{yz}(r)}{s_z^2(r)}, \quad s_{yz}(r) = \frac{1}{(r-1)} \left[\sum_{i=1}^{r} (y_i - \bar{y}_r)(z_i - \bar{z}_r) \right] \text{ and } s_z^2(r) = \frac{1}{r-1} \left[\sum_{i=1}^{r} (z_i - \bar{z}_r)^2 \right].$$

The point estimator of $\bar{Y}$ based on the imputation technique given in equation (2.1) is

$$T_{1u} = \frac{1}{u} \sum_{i \epsilon s_u} y_{.i} = \frac{1}{u} \left[\sum_{i \epsilon R_u} y_{.i} + \sum_{i \epsilon R_u^c} y_{.i} \right],$$

which results in $T_{1u} = \bar{y}_r + \hat{b}_1(\bar{Z} - \bar{z}_r)$.

$$(ii) \qquad y_{.i} = \begin{cases} y_i & \text{if } i \in R_u \\ \bar{y}_r + \hat{b}_2\left[\frac{u(\bar{Z}-\bar{z}_r)}{u-r} + z_i - \bar{z}_r\right] & \text{if } i \in R_u^c \end{cases} \tag{2.2}$$

where

$$\hat{b}_2 = \frac{s_{yz}(r)}{s_z^2(u)} \text{ and } s_z^2(u) = \frac{1}{u-1}\left[\sum_{i=1}^{u}(z_i - \bar{z}_u)^2\right].$$

The point estimator of $\bar{Y}$ based on the imputation technique given in equation (2.2) is derived as $T_{2u} = \bar{y_r} + \hat{b_2}(\bar{Z} - \bar{z_r})$.

$$(iii) \qquad y_{.i} = \begin{cases} y_i(\frac{\bar{Z}}{\bar{z}_u}) & \text{if } i \in R_u \\ \delta(\frac{\bar{Z}}{\bar{z}_u} z_i) & \text{if } i \in R_u^c \end{cases} \tag{2.3}$$

where $\delta = \frac{\sum_{i \in R_u} y_i}{\sum_{i \in R_u} z_i} = \bar{y}_r / \bar{z}_r$. The point estimator of $\bar{Y}$ based on the imputation technique given in equation (2.3) is obtained as $T_{3u} = (\bar{y}_r / \bar{z}_r)\bar{Z}$.

Estimators T_{1m} and T_{2m} of the set T_m are structured to estimate the population mean, utilizing the information on an auxiliary character z and the information available from the previous occasion as well. The estimators T_{1m} and T_{2m}, which are based on the matched sample s_m are defined as:

$$T_{1m} = (\bar{y}_m^* / \bar{x}_m^*)\bar{x}_n^* \text{ and } T_{2m} = (\bar{y}_m^{**} / \bar{x}_m^*)\bar{x}_n^*, \tag{2.4}$$

where $\bar{y}_m^* = \bar{y}_m + b_{yz}(m)(\bar{Z} - \bar{z}_m)$, $\bar{x}_m^* = \bar{x}_m + b_{xz}(m)(\bar{Z} - \bar{z}_m)$, $\bar{x}_n^* = \bar{x}_n + b_{xz}(m)(\bar{Z} - \bar{z}_n)$ and $\bar{y}_m^{**} = \bar{y}_m + b_{yx}(m)(\bar{x}_n - \bar{x}_m)$, and $b_{yz}(m), b_{xz}(m), b_{xz}(n)$ and $b_{yx}(m)$ are the sample regression coefficients between the variables shown in suffices and based on the sample sizes shown in braces.

Considering the convex linear combination of the estimators of sets T_u and T_m, we have the following sequence of estimators of population mean $\bar{Y}$ at second (current) occasion:

$$T_{ij} = \varphi_{ij} T_{iu} + (1 - \varphi_{ij}) T_{jm}, \tag{2.5}$$

where $\varphi_{ij} (i = 1, 2, 3; j = 1, 2)$ are the unknown constants to be determined under certain criterion.

Remark 1. For estimating the mean on each occasion the estimator $T_{iu} \quad (i = 1, 2, 3)$ is suitable, which implies that more belief on $T_{iu} \quad (i = 1, 2, 3)$ could be shown by choosing $\varphi_{ij} \quad (i = 1, 2, 3; j = 1, 2)$ as 1 (or close to 1), while for estimating the change from one occasion to the next, the estimator $T_{jm} \quad (j = 1, 2)$ could be more useful so $\varphi_{ij} \quad (i = 1, 2, 3; j = 1, 2)$ might be chosen as 0 (or close to 0). For asserting both the problems simultaneously, the suitable (optimum) choice of $\varphi_{ij} \quad (i = 1, 2, 3; j = 1, 2)$ is required.

3 Properties of the Estimators

Since, $T_{iu}\ \ (i = 1, 2, 3)$ are simple linear regression, ratio or chain-type ratio and regression estimators, they are biased for population mean $\bar{Y}$. Therefore, the resulting sequence of estimators $T_{ij}\ \ (i = 1, 2, 3; j = 1, 2)$ defined in equation is also biased estimator of $\bar{Y}$. The bias $B(.)$ and mean square error $M(.)$ up-to the first order of approximations of $T_{ij}\ \ (i = 1, 2, 3; j = 1, 2)$ are derived using large sample approximations given below:

$$\begin{aligned}
&\bar{y}_r = (1+e_1)\bar{Y}, \quad \bar{y}_m = (1+e_2)\bar{Y}, \quad \bar{x}_m = (1+e_3)\bar{X}, \bar{x}_n = (1+e_4)\bar{X}, \quad \bar{z}_r = (1+e_5)\bar{Z},\\
&\bar{z}_m = (1+e_6)\bar{Z}, \quad \bar{z}_n = (1+e_7)\bar{Z}, \quad s_{yz}(r) = (1+e_8)S_{yz}, s_{yz}(m) = (1+e_9)S_{yz},\\
&s_{xz}(m) = (1+e_{10})S_{xz}, \quad s_{xz}(n) = (1+e_{11})S_{xz}, \quad s_{yx}(m) = (1+e_{12})S_{yx},\\
&s_z^2(r) = (1+e_{13})S_z^2, \quad s_z^2(m) = (1+e_{14})S_z^2, \quad s_z^2(n) = (1+e_{15})S_z^2, \quad s_x^2(m) = (1+e_{16})S_x^2,\\
&s_z^2(u) = (1+e_{17})S_z^2; \quad \text{such that} \quad E(e_i) = 0 \text{ and } |e_i| < 1 \forall\ \ i = 1, \ldots, 17.
\end{aligned}$$

Under the above transformations we obtain

$$\left.\begin{aligned}
T_{1u} &= (1+e_1)\bar{Y} - (e_5 + e_5e_8 - e_5e_{13})\beta_{yz}\bar{Z}\\
T_{2u} &= (1+e_1)\bar{Y} - (e_5 + e_5e_8 - e_5e_{17})\beta_{yz}\bar{Z}\\
T_{3u} &= (1+e_1 - e_5 - e_1e_5 + e_5^2)\bar{Y}\\
T_{1m} &= \big\{(1+e_2)\bar{Y} - (e_6 + e_6e_9 - e_6e_{14})\beta_{yz}\bar{Z}\big\}\big\{(1+e_4) - (e_7 + e_7e_{11} - e_7e_{15})\\
&\quad \tfrac{\beta_{xz}}{\bar{X}}\bar{Z}\big\}\big[1 + \{e_3 - (e_6 + e_6e_{10} - e_6e_{14})\tfrac{\beta_{xz}}{\bar{X}}\bar{Z}\big]^{-1}\\
T_{2m} &= (1+e_2 - e_6 - e_2e_6 + e_6^2)\bar{Y} + (e_4 - e_3 + e_4e_{12} - e_3e_{12} - e_4e_{16}\\
&\quad + e_3e_{16} - e_4e_6 + e_3e_6)\beta_{yx}\bar{X}
\end{aligned}\right\} \tag{3.1}$$

Thus, we have the following theorems:

Theorem 1. *Bias of the sequence of estimators $T_{ij}\ \ (i = 1, 2, 3; j = 1, 2)$ to the first order of approximations is obtained as*

$$B(T_{ij}) = \varphi_{ij}B(T_{ij}) + (1-\varphi_{ij})B(T_{ij}); (i = 1, 2, 3; j = 1, 2), \tag{3.2}$$

where

$$\begin{aligned}
B(T_{1u}) &= \left(\frac{1}{r} - \frac{1}{N}\right)\left(\frac{\alpha_{011}\alpha_{003}}{\alpha_{002}^2} - \frac{\alpha_{012}}{\alpha_{002}}\right), \quad B(T_{2u}) = \left(\frac{1}{u} - \frac{1}{N}\right)\left(\frac{\alpha_{011}\alpha_{003}}{\alpha_{002}^2}\right) - \left(\frac{1}{r} - \frac{1}{N}\right)\left(\frac{\alpha_{012}}{\alpha_{002}}\right)\\
B(T_{3u}) &= \left(\frac{1}{r} - \frac{1}{N}\right)\left(\frac{\bar{Y}}{\bar{Z}^2}\alpha_{002} - \frac{\alpha_{011}}{\bar{Z}}\right)\\
B(T_{1m}) &= \left(\frac{1}{m} - \frac{1}{N}\right)\left(\frac{\alpha_{011}\alpha_{003}}{\alpha_{002}^2} - \frac{\alpha_{012}}{\alpha_{002}}\right) + \left(\frac{1}{m} - \frac{1}{n}\right)\frac{\bar{Y}}{\bar{X}}\\
&\quad \times \left[\left(\frac{\alpha_{200}}{\bar{X}} - \frac{\alpha_{110}}{\bar{Y}}\right) + \frac{\alpha_{101}}{\alpha_{002}}\left(\frac{\alpha_{011}}{\bar{Y}} - \frac{\alpha_{101}}{\bar{X}} + \frac{\alpha_{102}}{\alpha_{101}} - \frac{\alpha_{003}}{\alpha_{002}}\right)\right]\\
B(T_{2m}) &= \left(\frac{1}{m} - \frac{1}{N}\right)\left(\frac{\bar{Y}}{\bar{Z}^2}\alpha_{002} - \frac{\alpha_{011}}{\bar{Z}}\right) + \left(\frac{1}{m} - \frac{1}{n}\right)\left(\frac{\alpha_{110}\alpha_{101}}{\bar{Z}\alpha_{200}} + \frac{\alpha_{110}\alpha_{300}}{\alpha_{200}^2} - \frac{\alpha_{210}}{\alpha_{200}}\right),
\end{aligned}$$

where $\alpha_{rst} = E[(x - \bar{X})^r (y - \bar{Y})^s (z - \bar{Z})^t]$; *(r, s, t are positive integers)*

Proof. The bias of the sequence of estimators T_{ij} $(i = 1, 2, 3; j = 1, 2)$ is given by

$$\begin{aligned} B(T_{ij}) &= \varphi_{ij} E(T_{iu} - \bar{Y}) + (1 - \varphi_{ij}) E(T_{jm} - \bar{Y}) \\ &= \varphi_{ij} B(T_{iu}) + (1 - \varphi_{ij}) B(T_{jm}), \end{aligned} \tag{3.3}$$

where $B(T_{iu}) = E(T_{iu} - \bar{Y})$ and $B(T_{jm}) = E(T_{jm} - \bar{Y})$. Substituting the values of T_{iu} $(i = 1, 2, 3)$, T_{jm} $(j = 1, 2)$ from equations (3.1) in the equation (3.3) and taking expectations up to $o(n^{-1})$, we have the expression for the bias of the sequence of estimators T_{ij} $(i = 1, 2, 3; j = 1, 2)$ as described in equation (3.6). □

Theorem 2. *Mean square error of sequence of estimators* T_{ij} $(i = 1, 2, 3; j = 1, 2)$ *to the first order of approximations is obtained as*

$$M(T_{ij}) = \varphi_{ij}^2 M(T_{iu}) + (1 - \varphi_{ij})^2 M(T_{jm}) + 2\varphi_{ij}(1 - \varphi_{ij}) C(T_{iu}, T_{jm}); \quad (i = 1, 2, 3; j = 1, 2) \tag{3.4}$$

where $M(T_{ij}) = E(T_{ij} - \bar{Y})^2$, $C(T_{ij}, T_{i'j'}) = E(T_{ij} - \bar{Y})(T_{i'j'} - \bar{Y})$, $i \neq i'$ *and*

$$M(T_{1u}) = \left(\frac{1}{r} - \frac{1}{N}\right)(1 - \rho_{yz}^2) S_y^2 \tag{3.5}$$

$$M(T_{2u}) = \left(\frac{1}{r} - \frac{1}{N}\right)(1 - \rho_{yz}^2) S_y^2 \tag{3.6}$$

$$\begin{aligned} M(T_{3u}) &= \left(\frac{1}{r} - \frac{1}{N}\right) 2(1 - \rho_{yz}) S_y^2 \\ M(T_{1m}) &= \left[\left(\frac{1}{m} - \frac{1}{N}\right)(1 - \rho_{yz}^2) + \left(\frac{1}{m} - \frac{1}{n}\right)(1 - 2\rho_{yx} - \rho_{xz}^2 + 2\rho_{xz}\rho_{yz})\right] S_y^2 \\ M(T_{2m}) &= \left[\left(\frac{1}{m} - \frac{1}{N}\right) 2(1 - \rho_{yz}) + \left(\frac{1}{m} - \frac{1}{n}\right)(2\rho_{xz}\rho_{yx} - \rho_{yx}^2)\right] S_y^2 \end{aligned}$$

$$\begin{aligned} C(T_{1u}, T_{1m}) &= -\frac{S_y^2}{N}(1 - \rho_{yz}^2) S_y^2, \quad C(T_{1u}, T_{2m}) = -\frac{S_y^2}{N}(1 - \rho_{yz}^2) S_y^2 \\ C(T_{2u}, T_{1m}) &= -\frac{S_y^2}{N}(1 - \rho_{yz}^2) S_y^2, \quad C(T_{2u}, T_{2m} = -\frac{S_y^2}{N}(1 - \rho_{yz}^2) S_y^2 \\ C(T_{3u}, T_{1m}) &= -\frac{S_y^2}{N}(1 - \rho_{yz}^2) S_y^2, \quad C(T_{3u}, T_{2m}) = -\frac{S_y^2}{N} 2(1 - \rho_{yz}) S_y^2 \end{aligned} \tag{3.7}$$

Proof. It is obvious that mean square errors of the sequence of estimators T_{ij} $(i = 1, 2, 3; j = 1, 2)$ is given by

$$\begin{aligned} M(T_{ij}) &= E[\varphi_{ij}(T_{iu} - \bar{Y}) + (1 - \varphi_{ij})(T_{jm} - \bar{Y})]^2 \\ &= \varphi_{ij}^2 M(T_{iu}) + (1 - \varphi_{ij})^2 M(T_{jm}) + 2\varphi_{ij}(1 - \varphi_{ij}) E[(T_{iu} - \bar{Y})(T_{jm} - \bar{Y})]. \end{aligned} \tag{3.8}$$

Using the expressions given in equation (3.1) and taking expectations up to $o(n^{-1})$, we have the expression of mean square error of the sequence of estimators T_{ij} $(i = 1, 2, 3; j = 1, 2)$ given in equation (3.4). □

Remark 2. From equations (3.5) and (3.6) it follows that up to the first order of approximation, the mean square error of the estimators T_{1u} and T_{2u} are equal, subsequently, the mean square errors of the estimators T_{i1} and T_{2j} $(i,j=1,2)$ are also equal.

3.1 Minimum Mean Square Error of the Sequence of Estimators T_{ij} $(i=1,2,3; j=1,2)$

Since, mean square error of T_{ij} $(i=1,2,3; j=1,2)$ in equation (3.4) is a function of unknown constant φ_{ij} $(i=1,2,3; j=1,2)$, therefore, it is minimized with respect to φ_{ij} and subsequently the optimum values of φ_{ij} $(i=1,2,3; j=1,2)$ is obtained as

$$\varphi_{ij,opt} = \frac{M(T_{jm}) - C(T_{iu},T_{jm})}{M(T_{iu}) + M(T_{jm}) - 2C(T_{iu},T_{jm})}; (i=1,2,3; j=1,2) \tag{3.9}$$

Now substituting the value of $\varphi_{ij,opt}$ $(i,j=1,2)$ in equation (3.4), we get the optimum mean square error of T_{ij} as

$$M(T_{ij})_{opt} = \frac{M(T_{iu})M(T_{jm}) - C(T_{iu},T_{jm})^2}{M(T_{iu}) + M(T_{jm}) - 2C(T_{iu},T_{jm})}; (i=1,2,3; j=1,2). \tag{3.10}$$

Further substituting the values of $M(T_{iu})$, $M(T_{jm})$ and $C(T_{iu},T_{jm})$ $(i=1,2,3; j=1,2)$ the simplified values of $M(T_{ij})_{opt}$ $(i=1,2,3; j=1,2)$ are shown below:

$$\left.\begin{aligned}
M(T_{11})_{opt} &= M(T_{21})_{opt} = \left[\frac{\mu_{11}^{*2}A_{10}+\mu_{11}^{*}A_9+A_8}{\mu_{11}^{*2}A_7+\mu_{11}^{*}A_6+A_1}\right]\frac{S_y^2}{n}\\
M(T_{12})_{opt} &= M(T_{22})_{opt} = \left[\frac{\mu_{12}^{*2}A_{15}+\mu_{12}^{*}A_{14}+A_{13}}{\mu_{12}^{*2}A_{12}+\mu_{12}^{*}A_{11}+A_1}\right]\frac{S_y^2}{n}\\
M(T_{31})_{opt} &= \left[\frac{\mu_{31}^{*2}A_{19}+\mu_{31}^{*}A_{18}+A_{13}}{\mu_{31}^{*2}A_{17}+\mu_{31}^{*}A_{16}+A_2}\right]\frac{S_y^2}{n}\\
M(T_{32})_{opt} &= \left[\frac{\mu_{32}^{*2}A_{24}+\mu_{32}^{*}A_{23}+A_{22}}{\mu_{32}^{*2}A_{20}+\mu_{32}^{*}A_{21}+A_2}\right]\frac{S_y^2}{n},
\end{aligned}\right\} \tag{3.11}$$

where $A_1=(1-\rho_{yz}^2), A_2=2(1-\rho_{yz}), A_3=1-2\rho_{yx}-\rho_{xz}^2+2\rho_{xz}\rho_{yz}, A_4=2\rho_{xz}\rho_{yx}-\rho_{yx}^2, A_5=A_1-A_2, A_6=(f_1-1)A_1, A_7=f_1A_3, A_8=(1-f)A_1^2, A_9=A_1A_3-fA_1A_6, A_{10}=-fA_1A_7, A_{11}=f_1(A_2+fA_5)-A_1, A_{12}=f_1(A_4-fA_5), A_{13}=(1-f)A_1A_2, A_{14}=A_1(A_4+fA_2)-ff_1A_1(A_2+fA_5), A_{15}=-fA_1A_{12}, A_{16}=f_1(A_1+fA_5)-A_2, A_{17}=f_1(A_3-fA_5), A_{18}=A_2A_3+fA_1\{A_2-f_1(A_2+fA_5)\}, A_{19}=ff_1(fA_1A_5-A_2A_3), A_{20}=f_1A_4, A_{21}=(f_1-1)A_2, A_{22}=(1-f)A_2^2, A_{23}=A_2(A_4-fA_{21}), A_{24}=-fA_2A_{20}, f=\frac{n}{N}$ and μ_{ij}^* $(i=1,3; j=1,2)$ are fractions of fresh sample at the current (second) occasion for the estimators T_{ij} $(i=1,3; j=1,2)$.

3.2 Optimum Replacement Policy

To determine the optimum values of μ_{ij}^* $(i=1,3; j=1,2)$ (fraction of samples to be taken afresh at second occasion) so that population mean $\bar{Y}$ may be estimated with the maximum precision, we minimize mean square errors of T_{ij} $(i=1,3; j=1,2)$ given in equations

(3.11) respectively with respect to $(i = 1,3; j = 1,2)$. This yields quadratic equations in μ^*_{ij} $(i = 1,3; j = 1,2)$, and respective solutions of μ^*_{ij} say $\hat{\mu}^*_{ij}$ $(i = 1,3; j = 1,2)$ are given below:

$$\left.\begin{array}{lll} Q_1\mu^{*2}_{11} + 2Q_2\mu^*_{11} + Q_3 = 0 & \Rightarrow & \hat{\mu}^*_{11} = \frac{-Q_2 \pm \sqrt{Q_2^2 - Q_1Q_3}}{Q_1} \\ Q_4\mu^{*2}_{12} + 2Q_5\mu^*_{12} + Q_6 = 0 & \Rightarrow & \hat{\mu}^*_{12} = \frac{-Q_5 \pm \sqrt{Q_5^2 - Q_4Q_6}}{Q_4} \\ Q_7\mu^{*2}_{31} + 2Q_8\mu^*_{31} + Q_9 = 0 & \Rightarrow & \hat{\mu}^*_{31} = \frac{-Q_8 \pm \sqrt{Q_8^2 - Q_7Q_9}}{Q_7} \\ Q_{10}\mu^{*2}_{32} + 2Q_{11}\mu^*_{32} + Q_{12} = 0 & \Rightarrow & \hat{\mu}^*_{32} = \frac{-Q_{11} \pm \sqrt{Q_{11}^2 - Q_{11}Q_{12}}}{Q_{10}} \end{array}\right\} \tag{3.12}$$

where $Q_1 = A_6A_{10} - A_7A_9, Q_1 = A_1A_{10} - A_7A_8, Q_3 = A_1A_9 - A_6A_8, Q_4 = A_{11}A_{15} - A_{12}A_{14}, Q_5 = A_1A_{15} - A_{12}A_{13}, Q_6 = A_1A_{14} - A_{11}A_{13}, Q_7 = A_{16}A_{19} - A_{17}A_{18}, Q_8 = A_2A_{19} - A_{13}A_{17}, Q_9 = A_2A_{18} - A_{13}A_{16}, Q_{10} = A_{21}A_{24} - A_{20}A_{23}, Q_{11} = A_2A_{24} - A_{20}A_{22}, Q_{12} = A_2A_{23} - A_{21}A_{22}$.

From equations (3.12) it is obvious that real values of $\hat{\mu}^*_{ij}$ $(i = 1,3; j = 1,2)$ exist iff, the quantities under square roots are greater than or equal to zero. For any combination of correlations ρ_{yx}, ρ_{xz} and ρ_{yz}, which satisfy the conditions of real solutions; two real values of $\hat{\mu}^*_{ij}$ $(i = 1,3; j = 1,2)$ are possible. Hence, while choosing the values of $\hat{\mu}^*_{ij}$, it should be remembered that $0 \le \hat{\mu}^*_{ij} \le 1$. All the other values of $\hat{\mu}^*_{ij}$ $(i = 1,3; j = 1,2)$ are inadmissible. Substituting the admissible values of $\hat{\mu}^*_{ij}$ say $\mu^{*(0)}_{ij}$ $(i = 1,3; j = 1,2)$ from equations (3.12) into equations (3.11) respectively, we have the optimum values of mean square errors of T_{ij} $(i = 1,3; j = 1,2)$, which are shown below:

$$M(T^0_{11})_{opt} = \left[\frac{\mu^{*(0)2}_{11}A_{10} + \mu^{*(0)}_{11}A_9 + A_8}{\mu^{*(0)2}_{11}A_7 + \mu^{*(0)}_{11}A_6 + A_1}\right]\frac{S^2_y}{n}, \; M(T^0_{12})_{opt} = \left[\frac{\mu^{*(0)2}_{12}A_{15} + \mu^{*(0)}_{12}A_{14} + A_{13}}{\mu^{*(0)2}_{12}A_{12} + \mu^{*(0)}_{12}A_{11} + A_1}\right]\frac{S^2_y}{n}$$

$$M(T^0_{31})_{opt} = \left[\frac{\mu^{*(0)2}_{31}A_{19} + \mu^{*(0)}_{31}A_{18} + A_{13}}{\mu^{*(0)2}_{31}A_{17} + \mu^{*(0)}_{31}A_{16} + A_2}\right]\frac{S^2_y}{n}, \; M(T^0_{32})_{opt} = \left[\frac{\mu^{*(0)2}_{32}A_{24} + \mu^{*(0)}_{32}A_{23} + A_{22}}{\mu^{*(0)2}_{32}A_{20} + \mu^{*(0)}_{32}A_{21} + A_2}\right]\frac{S^2_y}{n}.$$

4 Comparisons and Conclusions

The percent relative loss in efficiencies of the estimators T_{ij} $(i = 1,3; j = 1,2)$ with respect to the estimators for the similar circumstances but under the complete response case (with no missing data) have been obtained to study the effect of non-response on the precision of estimates under two-occasion successive sampling. Estimators τ_{kj} $(k, j = 1,2)$ are defined under the same circumstances as the estimators T_{ij} $(i = 1,3; j = 1,2)$, but in the absence of non-response and shown as

$$\tau_{kj} = \psi_{kj}\tau_{ku} + (1 - \psi_{kj})T_{jm}; (k, j = 1,2)$$

where $\tau_{1u} = \bar{y}_u + b_{yz}(u)(\bar{Z} - \bar{z}_u)$, $\tau_{2u} = \frac{\bar{y}_u}{\bar{z}_u}\bar{Z}$ and T_{jm} $(j = 1,2)$ are defined in equations (2.4). ψ_{kj} $(k, j = 1,2)$ are unknown constants to be determined by the minimization of the mean square errors of τ_{kj} $(k, j = 1,2)$. Following the methods discussed in sections 4 and

5, the optimum mean square error of $\tau_{kj}\ \ (k, j = 1, 2)$ are given by

$$M(\tau_{11}^0)_{opt} = B_1\left[\frac{B_1 + \mu_{11}^{(0)}B_2}{B_1 + \mu_{11}^{(0)2}B_2} - f\right]\frac{S_y^2}{n} \qquad M(\tau_{12}^0)_{opt}\left[= \frac{B_{11} + \mu_{12}^{(0)}B_{10} + \mu_{12}^{(0)2}B_9}{B_1 + \mu_{12}^{(0)}B_7 + \mu_{12}^{(0)2}B_8}\right]\frac{S_y^2}{n}$$

$$M(\tau_{21}^0)_{opt} = \left[\frac{B_{11} + \mu_{21}^{(0)}B_{12} + \mu_{21}^{(0)2}B_{13}}{B_3 + \mu_{21}^{(0)}B_{14} + \mu_{21}^{(0)2}B_{15}}\right]\frac{S_y^2}{n} \qquad M(\tau_{22}^0)_{opt} = B_3\left[\frac{B_3 + \mu_{22}^{(0)}B_4}{B_3 + \mu_{22}^{(0)2}B_4} - f\right]\frac{S_y^2}{n}$$

and optimum values of $\mu_{ij}^{(0)}\ \ (i, j = 1, 2)$ are given by

$$\mu_{11}^{(0)} = \frac{-B_2 \pm \sqrt{B_2^2 + B_1B_3}}{B_1} \qquad \mu_{12}^{(0)} = \frac{-P_2 \pm \sqrt{P_2^2 - P_1P_3}}{P_1}$$

$$\mu_{21}^{(0)} = \frac{-P_5 \pm \sqrt{P_5^2 - P_4P_6}}{P_4} \qquad \mu_{22}^{(0)} = \frac{-B_3 \pm \sqrt{B_3^2 + B_3B_4}}{B_3}$$

where $B_1 = 1 - \rho_{yz}^2$, $B_2 = 1 - 2\rho_{yx} - \rho_{xz}^2 + 2\rho_{xz}\rho_{yz}$, $B_3 = 2(1 - \rho_{yz})$, $B_4 = 2\rho_{xz}\rho_{yx} - \rho_{yx}^2$, $B_5 = B_1 - B_3$, $B_6 = B_3 + fB_5$, $B_7 = -(1-f)B_5$, $B_8 = B_4 - fB_5$, $B_9 = -fB_1B_8$, $B_{10} = (B_4 - f^2B_5)B_1$, $B_{11} = (1-f)B_1B_3$, $B_{12} = f^2B_1B_5 + B_2B_3$, $B_{13} = f(fB_1B_5 - B_2B_3)$, $B_{14} = (1+f)B_5$, $B_{15} = B_2 - fB_5$, $P_1 = B_7B_9 - B_8B_{10}$, $P_2 = B_1B_9 - B_8B_{11}$, $P_3 = B_1B_{10} - B_7B_{11}$, $P_4 = B_{14}B_{13} - B_{12}B_{15}$, $P_5 = B_3B_{13} - B_{11}B_{15}$, $P_6 = B_3B_{12} - B_{11}B_{14}$ and $f = \frac{n}{N}$

Remark 3. To compare the performance of the estimators $T_{ij}\ \ (i = 1, 3; j = 1, 2)$ and $\tau_{kj}\ \ (k, j = 1, 2)$, we introduce an assumption $\rho_{xz} = \rho_{yz}$, which is an intuitive assumption, considered, for example by Cochran (1977) and Feng and Zou (1997).

The percent relative losses in precision of T_{ij} (i = 1, 3; j = 1, 2) with respect to τ_{kj} (k, j = 1, 2) under their respective optimality conditions are given by

$$L_1 = \frac{M(T_{11}^0)_{opt} - M(\tau_{11}^0)_{opt}}{M(T_{11}^0)_{opt}} \times 100, \qquad L_2 = \frac{M(T_{12}^0)_{opt} - M(\tau_{12}^0)_{opt}}{M(T_{12}^0)_{opt}} \times 100$$

$$L_3 = \frac{M(T_{31}^0)_{opt} - M(\tau_{21}^0)_{opt}}{M(T_{31}^0)_{opt}} \times 100, \quad \text{and} \quad L_4 = \frac{M(T_{32}^0)_{opt} - M(\tau_{22}^0)_{opt}}{M(T_{32}^0)_{opt}} \times 100.$$

For $N = 5000$, $n = 500$ and different choices of ρ_{yx} and ρ_{yz}, Tables 1-4 give the optimum values of $\mu_{kj}^{(0)}\ \ (k, j = 1, 2)$, $\mu_{ij}^{*(0)}\ \ (i = 1, 3; j = 1, 2)$ and percent relative loss $L_i\ \ (i = 1, 2, 3, 4)$ in precision of $T_{ij}\ \ (i = 1, 3; j = 1, 2)$ with respect to $\tau_{kj}\ \ (k, j = 1, 2)$.

The following conclusions can be read out from Tables 1-4:
From Table 1 it is clear that:

(a) For the fixed value of ρ_{yx} and ρ_{yz} the value of L_1 increases with the increasing values of t whereas no definite patterns are visible in $\mu_{11}^{*(0)}$. This phenomenon is obvious since, the higher the non-response rate, the higher the loss in precision occurs.

(b) For the fixed value of t and ρ_{yx}, the loss in precision L_1 decreases with increasing values of ρ_{yz} but the values of are increasing for some choices of ρ_{yz} and decreasing for few choices of ρ_{yz}. This behavior is highly desirable, since, it concludes that if highly correlated auxiliary character is available it pays in terms of enhance precision of estimates.

From Table 2 it is observed that:

(a) For the fixed values of ρ_{yx} and ρ_{yz} the values of $\mu_{12}^{*(0)}$ and L_2 increase with the increasing values of t which shows that the higher the non-response rate, the larger fresh sample is required at current occasion.

(b) For the fixed values of t and ρ_{yx}, no definite trends are seen in the values of $\mu_{12}^{*(0)}$ but the values of L_2 increases with the increase in the values of ρ_{yz}.

From Table 3 it can be seen that:

(a) For the fixed values of ρ_{yx} and ρ_{yz} the values of the behavior of $\mu_{31}^{*(0)}$ and L_3 are same as that of Table 1 when the value of t is increased.

(b) For the fixed values of t and ρ_{yx} the values of $\mu_{31}^{*(0)}$ and L_3 decreases with the increasing value of ρ_{yz}. This phenomenon indicates that if the correlation between the study character and the auxiliary character is high it not only enhances the precision of estimates but also reduces the cost of survey.

From Table 4 it is observed that:

(a) For the fixed values of ρ_{yx} and ρ_{yz} the patterns of $\mu_{32}^{*(0)}$ and L_4 are similar as Table 1 when the value of t is increased.

(b) For the fixed values of t and ρ_{yx} the values of $\mu_{32}^{*(0)}$ and L_4 decreases with the increasing values of ρ_{yz} which is highly desirable.

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Table 1: Percent relative loss L_1 in precision of T_{11} with respect to τ_{11}

ρ_{yz}		0.3			0.5			0.7			0.9		
t	ρ_{yx}	$\mu_{11}^{(0)}$	$\mu_{11}^{*(0)}$	L_1	$\mu_{11}^{(0)}$	$\mu_{11}^{*(0)}$	L_1	$\mu_{11}^{(0)}$	$\mu_{11}^{*(0)}$	L_1	$\mu_{11}^{(0)}$	$\mu_{11}^{*(0)}$	L_1
0.1	0.3	0.44	0.57	6.20	0.42	0.51	5.97	0.37	0.43	5.74	0.27	0.29	5.49
	0.5	-	*	-	0.46	0.65	6.55	0.41	0.49	5.93	0.30	0.33	5.55
	0.7	0.55	0.42	4.95	0.52	0.28	4.32	0.48	0.81	7.34	0.36	0.40	5.69
	0.9	0.68	0.65	5.61	0.66	0.62	5.55	0.61	0.56	5.41	-	*	-
0.2	0.3	0.44	0.72	13.96	0.42	0.61	13.05	0.37	0.49	12.22	0.27	0.32	11.43
	0.5	-	*	-	0.46	0.87	15.38	0.41	0.59	12.89	0.30	0.37	11.61
	0.7	0.55	0.27	8.45	0.52	0.00	5.86	-	*	-	0.36	0.46	12.04
	0.9	0.68	0.61	10.84	0.66	0.57	10.68	0.61	0.49	10.21	-	*	-
0.3	0.3	0.44	0.89	23.52	0.42	0.73	21.42	0.37	0.56	19.57	0.27	0.35	17.91
	0.5	-	*	-	-	*	-	0.41	0.70	21.08	0.30	0.41	18.28
	0.7	0.55	0.08	10.26	-	*	-	-	*	-	0.36	0.52	19.18
	0.9	0.68	0.56	15.66	0.66	0.52	15.32	0.61	0.41	14.29	-	*	-
0.4	0.3	-	*	-	0.42	0.88	31.34	0.37	0.65	27.95	0.27	0.39	25.04
	0.5	-	*	-	-	*	-	0.41	0.84	30.70	0.30	0.46	25.68
	0.7	-	*	-	-	*	-	-	*	-	0.36	0.59	27.26
	0.9	0.68	0.50	19.99	0.66	0.45	19.40	0.61	0.31	17.56	-	*	-
0.5	0.3	-	*	-	-	*	-	0.37	0.77	37.64	0.27	0.44	33.00
	0.5	-	*	-	-	*	-	-	*	-	0.30	0.52	33.99
	0.7	-	*	-	-	*	-	-	*	-	0.36	0.70	36.52
	0.9	0.68	0.43	23.76	0.66	0.37	22.81	0.61	0.19	19.81	-	*	-
0.6	0.3	-	*	-	-	*	-	0.37	0.93	48.99	0.27	0.51	42.03
	0.5	-	*	-	-	*	-	-	*	-	0.30	0.61	43.50
	0.7	-	*	-	-	*	-	-	*	-	0.36	0.84	47.30
	0.9	0.68	0.33	26.81	0.66	0.25	25.37	0.61	0.02	20.76	-	*	-

"$\star$" in the tables indicate that the admissible values of $\mu_{ij}^{*(0)}$ $(i = 1, 3; j = 1, 2)$ do not exist

Table 2: Percent relative loss L_2 in precision of T_{12} with respect to τ_{12}

ρ_{yz}		0.3			0.5			0.7			0.9		
t	ρ_{yx}	$\mu_{12}^{(0)}$	$\mu_{12}^{*(0)}$	L_2	$\mu_{12}^{(0)}$	$\mu_{12}^{*(0)}$	L_2	$\mu_{12}^{(0)}$	$\mu_{12}^{*(0)}$	L_2	$\mu_{12}^{(0)}$	$\mu_{12}^{*(0)}$	L_2
0.1	0.3	0.15	0.31	2.05	0.29	0.39	3.84	0.35	0.42	4.78	0.34	0.38	5.20
	0.5	0.20	0.33	2.75	0.31	0.39	4.20	0.34	0.39	4.91	0.31	0.34	5.21
	0.7	0.20	0.33	2.75	0.31	0.39	4.28	0.33	0.39	4.95	0.29	0.33	5.20
	0.9	0.15	0.31	2.05	0.31	0.39	4.20	0.33	0.39	4.95	0.29	0.32	5.20
0.2	0.3	0.15	0.49	4.08	0.29	0.53	7.65	0.35	0.51	9.52	0.34	0.43	10.36
	0.5	0.20	0.48	5.48	0.31	0.49	8.38	0.34	0.46	9.79	0.31	0.38	10.38
	0.7	0.20	0.48	5.48	0.31	0.49	8.54	0.33	0.45	9.87	0.29	0.36	10.38
	0.9	0.15	0.49	4.08	0.31	0.49	8.38	0.33	0.45	9.87	0.29	0.35	10.38
0.3	0.3	0.15	0.72	6.12	0.29	0.68	11.44	0.35	0.61	14.23	0.34	0.49	15.49
	0.5	0.20	0.66	8.20	0.31	0.62	12.52	0.34	0.55	14.64	0.31	0.43	15.52
	0.7	0.20	0.66	8.20	0.31	0.60	12.76	0.33	0.52	14.76	0.29	0.41	15.52
	0.9	0.15	0.72	6.12	0.31	0.62	12.52	0.33	0.52	14.76	0.29	0.40	15.52
0.4	0.3	0.15	0.99	8.14	0.29	0.88	15.20	0.35	0.74	18.91	0.34	0.56	20.58
	0.5	0.20	0.88	10.91	0.31	0.77	16.64	0.34	0.65	19.45	0.31	0.49	20.62
	0.7	0.20	0.88	10.91	0.31	0.74	16.96	0.33	0.61	19.61	0.29	0.46	20.63
	0.9	0.15	0.99	8.14	0.31	0.77	16.64	0.33	0.61	19.61	0.29	0.45	20.63
0.5	0.3	-	*	-	-	*	-	0.35	0.91	23.55	0.34	0.66	25.63
	0.5	-	*	-	0.31	0.97	20.73	0.34	0.78	24.22	0.31	0.57	25.69
	0.7	-	*	-	0.31	0.93	21.13	0.33	0.73	24.42	0.29	0.53	25.70
	0.9	-	*	-	0.31	0.97	20.73	0.33	0.73	24.42	0.29	0.52	25.70
0.6	0.3	-	*	-	-	*	-	-	*	-	0.34	0.79	30.65
	0.5	-	*	-	-	*	-	0.34	0.95	28.96	0.31	0.67	30.73
	0.7	-	*	-	-	*	-	0.33	0.89	29.20	0.29	0.62	30.75
	0.9	-	*	-	-	*	-	0.33	0.89	29.20	0.29	0.61	30.75

"$\star$" in the tables indicate that the admissible values of $\mu_{ij}^{*(0)}$ $(i = 1, 3; j = 1, 2)$ do not exist

Table 3: Percent relative loss L_3 in precision of T_{31} with respect to τ_{21}

ρ_{yz}		0.3			0.5			0.7			0.9		
t	ρ_{yx}	$\mu_{21}^{(0)}$	$\mu_{31}^{*(0)}$	L_3	$\mu_{21}^{(0)}$	$\mu_{31}^{*(0)}$	L_3	$\mu_{21}^{(0)}$	$\mu_{31}^{*(0)}$	L_3	$\mu_{21}^{(0)}$	$\mu_{31}^{*(0)}$	L_3
0.1	0.3	-	*	-	0.64	0.73	7.896	0.45	0.50	6.339	0.28	0.30	5.559
	0.5	-	*	-	-	*	-	0.53	0.61	6.911	0.32	0.34	5.647
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.42	5.848
	0.9	0.55	0.51	3.488	0.56	0.52	3.995	0.54	0.47	4.280	-	*	-
0.2	0.3	-	*	-	0.64	0.85	16.89	0.45	0.57	13.41	0.28	0.33	11.57
	0.5	-	*	-	-	*	-	0.53	0.71	14.85	0.32	0.38	11.80
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.48	12.36
	0.9	0.55	0.46	6.631	0.56	0.46	7.575	0.54	0.40	7.946	-	*	-
0.3	0.3	-	*	-	0.64	0.98	27.18	0.45	0.65	21.35	0.28	0.36	18.12
	0.5	-	*	-	-	*	-	0.53	0.83	23.99	0.32	0.42	18.56
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.54	19.65
	0.9	0.55	0.39	9.386	0.56	0.39	10.69	0.54	0.31	10.91	-	*	-
0.4	0.3	-	*	-	-	*	-	0.45	0.75	30.32	0.28	0.41	25.33
	0.5	-	*	-	-	*	-	0.53	0.98	34.56	0.32	0.48	26.05
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.62	27.89
	0.9	0.55	0.33	11.69	0.56	0.32	13.26	0.54	0.20	13.05	-	*	-
0.5	0.3	-	*	-	-	*	-	0.45	0.87	40.58	0.28	0.46	33.35
	0.5	-	*	-	-	*	-	-	*	-	0.32	0.54	34.46
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.73	37.30
	0.9	0.55	0.23	13.45	0.56	0.22	15.16	0.54	0.06	14.18	-	*	-
0.6	0.3	-	*	-	-	*	-	-	*	-	0.28	0.53	42.46
	0.5	-	*	-	-	*	-	-	*	-	0.32	0.64	44.06
	0.7	-	*	-	-	*	-	-	*	-	0.38	0.87	48.23
	0.9	0.55	0.10	14.52	0.56	0.08	16.23	-	*	-	-	*	-

"$\star$" in the tables indicate that the admissible values of $\mu_{ij}^{*(0)}$ $(i = 1, 3; j = 1, 2)$ do not exist

Table 4: Percent relative loss L_4 in precision of T_{32} with respect to τ_{22}

ρ_{yz}		0.3			0.5			0.7			0.9		
t	ρ_{yx}	$\mu_{22}^{(0)}$	$\mu_{32}^{*(0)}$	L_4	$\mu_{22}^{(0)}$	$\mu_{32}^{*(0)}$	L_4	$\mu_{22}^{(0)}$	$\mu_{32}^{*(0)}$	L_4	$\mu_{22}^{(0)}$	$\mu_{32}^{*(0)}$	L_4
0.1	0.3	0.46	0.65	6.544	0.44	0.57	6.189	0.42	0.49	5.922	0.36	0.40	5.679
	0.5	0.45	0.60	6.336	0.43	0.52	6.030	0.39	0.45	5.807	0.33	0.36	5.602
	0.7	0.45	0.60	6.336	0.43	0.51	5.993	0.38	0.44	5.769	0.31	0.34	5.571
	0.9	0.46	0.65	6.544	0.43	0.52	6.030	0.38	0.44	5.769	0.31	0.34	5.562
0.2	0.3	0.46	0.87	15.34	0.44	0.71	13.91	0.41	0.58	12.87	0.36	0.45	12.01
	0.5	0.45	0.78	14.50	0.43	0.64	13.29	0.39	0.52	12.45	0.33	0.40	11.76
	0.7	0.45	0.78	14.50	0.42	0.62	13.14	0.38	0.50	12.32	0.31	0.38	11.67
	0.9	0.46	0.87	15.34	0.43	0.64	13.29	0.38	0.50	12.32	0.31	0.37	11.64
0.3	0.3	-	*	-	0.44	0.89	23.41	0.41	0.69	21.03	0.36	0.51	19.11
	0.5	0.45	0.99	24.77	0.43	0.77	21.97	0.39	0.61	20.08	0.33	0.45	18.59
	0.7	0.45	0.99	24.77	0.42	0.75	21.64	0.38	0.58	19.78	0.31	0.42	18.39
	0.9	-	*	-	0.43	0.77	21.97	0.38	0.58	19.78	0.31	0.41	18.34
0.4	0.3	-	*	-	-	*	-	0.41	0.83	30.62	0.36	0.59	27.14
	0.5	-	*	-	0.43	0.94	32.35	0.39	0.72	28.89	0.33	0.51	26.22
	0.7	-	*	-	0.42	0.90	31.74	0.38	0.68	28.33	0.31	0.48	25.87
	0.9	-	*	-	0.43	0.94	32.35	0.38	0.68	28.33	0.31	0.47	25.77
0.5	0.3	-	*	-	-	*	-	-	*	-	0.36	0.69	36.33
	0.5	-	*	-	-	*	-	0.39	0.86	39.15	0.33	0.59	34.85
	0.7	-	*	-	-	*	-	0.38	0.80	38.25	0.31	0.55	34.29
	0.9	-	*	-	-	*	-	0.38	0.80	38.25	0.31	0.54	34.14
0.6	0.3	-	*	-	-	*	-	-	*	-	0.36	0.82	47.01
	0.5	-	*	-	-	*	-	-	*	-	0.33	0.69	44.78
	0.7	-	*	-	-	*	-	0.38	0.97	49.92	0.31	0.64	43.95
	0.9	-	*	-	-	*	-	0.38	0.97	49.92	0.31	0.63	43.73

"$\star$" in the tables indicate that the admissible values of $\mu_{ij}^{*(0)}$ $(i = 1, 3; j = 1, 2)$ do not exist

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PRELIMINARY TEST ESTIMATION OF THE MEAN VECTOR UNDER BALANCED LOSS FUNCTION

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SUMMARY

In this paper we consider the so-called preliminary test approach under Bayesian setup. A class of shrinkage Bayes estimators is constructed and its performance is investigated under balanced loss function for some special members, with focus on preliminary test estimator. Risk analysis is then added to compare the performance of unrestricted and restricted Bayes estimators with the preliminary test estimator under balanced loss function.

Keywords and phrases: Balanced loss function, Bayes estimator, Normal-inverted Wishart, Preliminary test estimator.

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1 Introduction

Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, Y_N$ be independent and identically distributed (iid) as $\mathcal{N}_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ where the mean vector $\boldsymbol{\theta}$ and the positive definite covariance matrix $\boldsymbol{\Sigma}$ are both unknown. When nothing is known about the mean vector $\boldsymbol{\theta}$, then the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$ as unrestricted estimator (UE) is given by

$$\bar{\boldsymbol{Y}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{Y}_i. \tag{1.1}$$

It is well documented that James and Stein (1961) and Efron and Morris (1972, 1976) considered a decision-theoretic approaches to the estimation of $\boldsymbol{\theta}$ while $\boldsymbol{\Sigma}$ is known and unknown respectively. More recently Srivastava and Saleh (2005) considered the estimation of $\boldsymbol{\theta}$ under subspace restriction for unknown $\boldsymbol{\Sigma}$. Also Saleh and Kibria (2009) investigated on some improved estimators of $\boldsymbol{\theta}$ parallel to the latter work under elliptical symmetry. All mentioned references took quadratic loss function into account to study the performance of the estimators. In this approach we study the behavior of some improved estimators upon UE under so called balanced loss function (BLF). Sanjari Farsipour and Asgharzadeh (2003) also derived the Bayes estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\Sigma}$ under BLF. Importance of any estimation problem is boosted if we can furnish our driven estimators with good performance in the sense of having smaller risk. In this case, the loss function under study plays deterministic role. However, selecting objective or subjective points of view changes the results, it is utterly important to take reasonable and practical losses into account.

Let $\boldsymbol{\theta}^*$ denote any estimator of $\boldsymbol{\theta}$; then the quadratic loss function which reflects the goodness of fit of the model is $(\boldsymbol{\theta}^* - \boldsymbol{Y})'(\boldsymbol{\theta}^* - \boldsymbol{Y})$ where $\boldsymbol{Y} = (\boldsymbol{Y}_1, \cdots, \boldsymbol{Y}_N)$. Similarly, the precision of estimation of $\boldsymbol{\theta}^*$ is measured by the loss function $(\boldsymbol{\theta}^* - \boldsymbol{\theta})'(\boldsymbol{\theta}^* - \boldsymbol{\theta})$. Generally, both of the previous criteria are used to judge the performance of any estimator. Throughout this paper, we shall consider the estimation problem through the following loss function

$$\begin{aligned} L^{\boldsymbol{W}}_{\omega,\boldsymbol{\theta}_0}(\boldsymbol{\theta}^*;\boldsymbol{\theta}) &= \omega r\left(\|\boldsymbol{\theta}\|^2\right)(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)'\boldsymbol{W}(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0) \\ &\quad +(1-\omega) r\left(\|\boldsymbol{\theta}\|^2\right)(\boldsymbol{\theta}^* - \boldsymbol{\theta})'\boldsymbol{W}(\boldsymbol{\theta}^* - \boldsymbol{\theta}), \end{aligned} \tag{1.2}$$

where $\omega \in [0,1]$, $r(.)$ is a positive weight function, $\boldsymbol{W}$ is a weight matrix, and $\boldsymbol{\theta}_0$ is a target estimator (natural estimator such as MLE and least squares estimator). This loss is pioneered by Jozani et al. (2006) inspiring by Zellner's (1994) balanced loss function. This loss function takes both goodness of fit and error of estimation into account. The $\omega r\left(\|\boldsymbol{\theta}\|^2\right)(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)'\boldsymbol{W}(\boldsymbol{\theta}^* - \boldsymbol{\theta}_0)$ part of the loss is analogous to a penalty term for lack of smoothness in nonparametric regression. The weight ω in (1.2) calibrates the relative importance of these two criteria. Dey et al. (1999) also considered issues of admissibility and dominance, under the loss (1.2) ignoring the term $r(.)$ when $\boldsymbol{W} = \boldsymbol{I}_p$. For the case $\omega = 0$, we will simply write $L^{\boldsymbol{W}}_0(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ as the quadratic loss function. Of course, duty of the weight function $r(.)$ is clearly apparent in deriving the Bayes risk. In this paper, we take it into consideration for the sake of generality.

Assume $\boldsymbol{h}_i : \mathbb{R}^p \to \mathbb{R}^p$, $i = 1, 2$ are measurable functions.

Lemma 1.1.

(i) The estimator $\boldsymbol{\theta}_0 + (1-\omega)\boldsymbol{h}_1$ dominates $\boldsymbol{\theta}_0 + (1-\omega)\boldsymbol{h}_2$ under the balanced loss function $L^{\boldsymbol{W}}_{\omega,\boldsymbol{\theta}_0}(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ if and only if $\boldsymbol{\theta}_0 + \boldsymbol{h}_1$ dominates $\boldsymbol{\theta}_0 + \boldsymbol{h}_2$ under the quadratic loss function $L^{\boldsymbol{W}}_0(\boldsymbol{\theta}^*;\boldsymbol{\theta})$.

(ii) Suppose the estimator $\boldsymbol{\theta}_0$ has constant risk γ under the quadratic loss function $L^{\boldsymbol{W}}_0(\boldsymbol{\theta}^*;\boldsymbol{\theta})$. Then $\boldsymbol{\theta}_0$ is minimax under the balanced loss function $L^{\boldsymbol{W}}_{\omega,\boldsymbol{\theta}_0}(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ with constant (and minimax) risk $(1-\omega)\gamma$ if and only if $\boldsymbol{\theta}_0$ is minimax under the quadratic loss function $L^{\boldsymbol{W}}_0(\boldsymbol{\theta}^*;\boldsymbol{\theta})$ with constant (and minimax) risk γ.

The proof is a direct consequence of Corollary 1 (b) and Theorem 1 of Jozani et al. (2006) under multivariate case.

The gist of this paper is the estimation of the regression vector-parameter $\boldsymbol{\theta}$ when it is suspected that $\boldsymbol{\theta}$ may belong to the sub-space defined by $\boldsymbol{\theta} = \boldsymbol{B}\boldsymbol{\eta}$ where $\boldsymbol{B}$ is a $p \times r$ matrix of known constants with rank r and $\boldsymbol{\eta} \in \mathbb{R}^r$ with focus on the preliminary test estimator (PTE). Recent book of Saleh (2006) presents an overview on the topic under normal as well as nonparametric theory covering many standard models.

2 Proposed Estimators

If we know that for the known matrix $\boldsymbol{B}$ of rank r and $\boldsymbol{\eta} \in \mathbb{R}^r$, the hypothesis

$$H_0 \quad : \quad \boldsymbol{\theta} = \boldsymbol{B}\boldsymbol{\eta} \tag{2.1}$$

holds, then the MLE of $\boldsymbol{\theta}$ to be denoted by $\hat{\boldsymbol{\theta}}$ as restricted estimator (RE) is given by

$$\hat{\boldsymbol{\theta}} = \boldsymbol{B}(\boldsymbol{B}'\boldsymbol{S}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{S}^{-1}\bar{\boldsymbol{Y}}, \tag{2.2}$$

where

$$\boldsymbol{S} = \sum_{i=1}^{N}(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})'. \tag{2.3}$$

(See Srivastava and Khatri, 1979 and Srivastava and Saleh, 2005.)
Now under Bayesian viewpoint, to determine the Bayes estimator of $\boldsymbol{\theta}$ under the loss (1.2), it is enough to find a value $\boldsymbol{\theta}^*$ which minimizes

$$\begin{aligned} E\left[L^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\boldsymbol{\theta}^*;\boldsymbol{\theta})|\boldsymbol{Y}\right] &= \omega E\left[r\left(\|\boldsymbol{\theta}\|^2\right)|\boldsymbol{Y}\right](\boldsymbol{\theta}^* - \bar{\boldsymbol{Y}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^* - \bar{\boldsymbol{Y}}) \\ &\quad +(1-\omega)E\left[r\left(\|\boldsymbol{\theta}\|^2\right)(\boldsymbol{\theta}^* - \boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}^* - \boldsymbol{\theta})|\boldsymbol{Y}\right], \end{aligned} \tag{2.4}$$

where $\bar{\boldsymbol{Y}}$ is given by (1.1). Differentiating from (2.4) with respect to (w.r.t.) $\boldsymbol{\theta}^*$ and setting the derivative equal to zero gives the Bayes estimator

$$\hat{\boldsymbol{\theta}}_B = \omega\bar{\boldsymbol{Y}} + (1-\omega)\,\frac{E\left[r\left(\|\boldsymbol{\theta}\|^2\right)\boldsymbol{\theta}|\boldsymbol{Y}\right]}{E\left[r\left(\|\boldsymbol{\theta}\|^2\right)|\boldsymbol{Y}\right]}. \tag{2.5}$$

Also as

$$\begin{aligned} \frac{\partial^2 E\left[L^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\boldsymbol{\theta}^*;\boldsymbol{\theta})|\boldsymbol{Y}\right]}{\partial\boldsymbol{\theta}^{*\prime}\partial\boldsymbol{\theta}^*} &= 2(1-\omega)\boldsymbol{\Sigma}^{-1}r\left(\|\boldsymbol{\theta}\|^2\right) + 2\omega\boldsymbol{\Sigma}^{-1}r\left(\|\boldsymbol{\theta}\|^2\right) \\ &= 2\boldsymbol{\Sigma}^{-1}r\left(\|\boldsymbol{\theta}\|^2\right), \end{aligned}$$

is a positive definite matrix, $\hat{\boldsymbol{\theta}}_B$ actually corresponds to a minimum value.

However, different weight functions $r(.)$ give various types of $\hat{\boldsymbol{\theta}}_B$; for the sake of simplicity we take $r\left(\|\boldsymbol{\theta}\|^2\right) = 1$ throughout (one other choice can be $r\left(\|\boldsymbol{\theta}\|^2\right) = \frac{a}{\boldsymbol{\theta}'\boldsymbol{\theta}}$, $a > 0$). Then we have

$$\hat{\boldsymbol{\theta}}_B = \omega\bar{\boldsymbol{Y}} + (1-\omega)E(\boldsymbol{\theta}|\boldsymbol{Y}). \tag{2.6}$$

Now consider the normal-inverted Wishart distribution as a prior of $(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, i.e,

$$\begin{aligned} h(\boldsymbol{\theta}, \boldsymbol{\Sigma}) \quad &\propto \quad |\boldsymbol{\Lambda}|^{-\frac{1}{2}} \exp\left[\frac{-1}{2b}(\boldsymbol{\theta} - \boldsymbol{B}\boldsymbol{\eta})'\boldsymbol{\Lambda}^{-1}(\boldsymbol{\theta} - \boldsymbol{B}\boldsymbol{\eta})\right] \\ &\quad \times|\boldsymbol{\Lambda}|^{-\frac{m}{2}} \exp\left[\frac{-1}{2}tr\boldsymbol{\Lambda}^{-1}\boldsymbol{Q}\right], \end{aligned} \tag{2.7}$$

where $b > 0$, $m > 2p$, $\boldsymbol{\Lambda} = N^{-1}\boldsymbol{\Sigma}$ and $\boldsymbol{Q}$ is a positive definite matrix.
Then the marginal posterior distribution of $\boldsymbol{\theta}$ is given by

$$p(\boldsymbol{\theta}|\bar{\boldsymbol{Y}}, \boldsymbol{S}) \propto \left\{1 + (\boldsymbol{\theta} - \boldsymbol{\theta}_*)'\boldsymbol{\Lambda}_*^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_*)\right\}^{-\frac{N+m-p}{2}},$$

where

$$\boldsymbol{\theta}_* = \frac{\bar{\boldsymbol{Y}} + b\boldsymbol{B\eta}}{1+b}, \quad \boldsymbol{\Lambda}_* = \frac{\boldsymbol{S} + \boldsymbol{Q}}{1+b} + \frac{b}{(1+b)^2}\,(\bar{\boldsymbol{Y}} - \boldsymbol{B\eta})(\bar{\boldsymbol{Y}} - \boldsymbol{B\eta})',$$

which is the multivariate Student's t-distribution.

Then the Bayes estimator $\hat{\boldsymbol{\theta}}_B$ given by (2.5) is rewritten as

$$\hat{\boldsymbol{\theta}}_B = \frac{1 + b\omega}{1+b}\,\bar{\boldsymbol{Y}} + \frac{(1-\omega)b}{1+b}\,\boldsymbol{B\eta}. \tag{2.8}$$

It is convenient to use the estimate of $\boldsymbol{\theta}$ given by (2.2) to obtain the empirical Bayes estimator (EBE) of $\boldsymbol{\theta}$ as a convex combination of $\bar{\boldsymbol{Y}}$ and $\hat{\boldsymbol{\theta}}$ namely,

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{EB}(b) &= \frac{1 + b\omega}{1+b}\,\bar{\boldsymbol{Y}} + \frac{(1-\omega)b}{1+b}\,\hat{\boldsymbol{\theta}} \\ &= \bar{\boldsymbol{Y}} - \frac{(1-\omega)b}{1+b}(\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\theta}}),\end{aligned} \tag{2.9}$$

where b is arbitrary and unknown. In the continuation we estimate b to determine the specific estimator of $\boldsymbol{\theta}$. For this, it can be investigated that $(1+b)^{-1}\mathcal{L}_N = F_{q,m}$, where

$$\begin{aligned}\mathcal{L}_N &= \frac{m}{q}\,N\bar{\boldsymbol{Y}}'\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}\boldsymbol{C})^{-1}\boldsymbol{C}'\bar{\boldsymbol{Y}} \stackrel{D}{\to} \frac{m}{q}\,T^2 \\ T^2 &= N\bar{\boldsymbol{Y}}'\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}\boldsymbol{C})^{-1}\boldsymbol{C}'\bar{\boldsymbol{Y}},\end{aligned}$$

$m = N - q$, $q = p - r$, $\stackrel{D}{\to}$ stands for "equal in distribution", $\boldsymbol{C}$ is a $p \times q$ matrix of rank q such that $\boldsymbol{C}'\boldsymbol{B} = \boldsymbol{0}$, and $F_{q,m}$ denotes the F-distribution with (q, m) degrees of freedom. Hence, $(1+b)^{-1}$ can be estimated by a scalar multiple of $\mathcal{L}_N^{-1}$. (See Srivastava and Saleh, 2005.)

More generally, let $g(\mathcal{L}_N)$, a real-valued function of $\mathcal{L}_N$, be an estimate of $\frac{b}{b+1}$. Then the estimator given by (2.9) becomes

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{EB}(\hat{b}) &= \bar{\boldsymbol{Y}} - (1-\omega)g(\mathcal{L}_N)(\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\theta}}) \\ &= \hat{\boldsymbol{\theta}} + (1-\omega)\left[1 - g(\mathcal{L}_N)\right](\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\theta}}).\end{aligned} \tag{2.10}$$

The estimator $\hat{\boldsymbol{\theta}}_{EB}(\hat{b})$ is somehow similar to that introduced by Srivastava and Saleh (2005), with an additional weight $(1-\omega)$. Thus of course the performance of $\hat{\boldsymbol{\theta}}_{EB}(\hat{b})$ for different selections of $g(\mathcal{L}_N)$ should be the same as that discussed in earlier work under quadratic loss function. However, it is worthwhile studying the performance of $\hat{\boldsymbol{\theta}}_{EB}(\hat{b})$ under BLF by the use of important role of Lemma 1.1.

Now consider the following three choices of $g(\mathcal{L}_N)$:

(i) $g(\mathcal{L}_N) = 0$, $\hat{\boldsymbol{\theta}}_1 = \hat{\boldsymbol{\theta}}_{EB}(\hat{b}) = \bar{\boldsymbol{Y}}$

(ii) $g(\mathcal{L}_N) = 1$, $\hat{\boldsymbol{\theta}}_2 = \hat{\boldsymbol{\theta}}_{EB}(\hat{b}) = \hat{\boldsymbol{\theta}}$

(iii) $g(\mathcal{L}_N) = I(\mathcal{L}_N \leq F_{q,m}(\alpha)) = I(T^2 \leq \frac{q}{m}\, F_{q,m}(\alpha))$ where $F_{q,m}(\alpha)$ is the upper $100\alpha\%$ point of the $F_{q,m}$ distribution

$$\hat{\boldsymbol{\theta}}_3 = \hat{\boldsymbol{\theta}}^{PT} = \hat{\boldsymbol{\theta}}_{EB}(\hat{b}) = \bar{\boldsymbol{Y}} - (1-\omega) I(T^2 \leq \frac{q}{m}\, F_{q,m}(\alpha))(\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\theta}})$$

which is the preliminary test estimator (PTE) introduced by Bancroft (1944).

3 Risk Analysis

The risk function for any estimator $\boldsymbol{\theta}^*$ of $\boldsymbol{\theta}$ associated with (1.2) is defined as

$$\boldsymbol{R}^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\boldsymbol{\theta}^*; \boldsymbol{\theta}) = E[L^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\boldsymbol{\theta}^*; \boldsymbol{\theta})]. \tag{3.1}$$

In this section, first we determine the risk function using (3.1). For the case $\omega = 0$, we will simply write $\boldsymbol{R}^{\boldsymbol{\Sigma}^{-1}}_0(\boldsymbol{\theta}^*; \boldsymbol{\theta})$. Then some comparative results are given.

Simply

$$\boldsymbol{R}^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\bar{\boldsymbol{Y}}; \boldsymbol{\theta}) = p\,(1-\omega). \tag{3.2}$$

By making use of (2.2), $\boldsymbol{H} = \boldsymbol{I}_p - \boldsymbol{B}(\boldsymbol{B}'\boldsymbol{S}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{S}^{-1}$ and the utilities in Srivastava and Saleh (2005), we get

$$\begin{aligned}
\boldsymbol{R}^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\hat{\boldsymbol{\theta}}; \boldsymbol{\theta}) &= (1-\omega) E\left[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})\right] + \omega E\left(\bar{\boldsymbol{Y}}'\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\bar{\boldsymbol{Y}}\right) \\
&= \frac{1-\omega}{N}\left(p - \frac{q(n-p-1)}{n-q-1} + \frac{n-q+r-1}{n-q-1}\,\Delta^2\right) \\
&\quad + \frac{\omega}{N}\left(q + \Delta^2 + \frac{rm(q+\Delta^2)}{q(m-2)}\right),
\end{aligned} \tag{3.3}$$

where $\Delta^2 = N\boldsymbol{\theta}'\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{\theta}$.

Finally for the risk of $\hat{\boldsymbol{\theta}}^{PT}$, using (3.3) and equation (3.12) of Srivastava and Saleh

(2005), we have

$$
\begin{aligned}
\boldsymbol{R}^{\boldsymbol{\Sigma}^{-1}}_{\omega,\bar{\boldsymbol{Y}}}(\hat{\boldsymbol{\theta}}^{PT};\boldsymbol{\theta}) &= (1-\omega)E\left[(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\hat{\boldsymbol{\theta}}^{PT}-\boldsymbol{\theta})\right] + \omega(1-\omega)^2 \\
&\times E\left[I(T^2 \le \frac{q}{m}\, F_{q,m}(\alpha))(\bar{\boldsymbol{Y}}-\hat{\boldsymbol{\theta}})'\boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{Y}}-\hat{\boldsymbol{\theta}})\right] \\
&+ \frac{1-\omega}{N}\Bigg\{p - q\Bigg[G_{q+2,m}\left(\frac{q}{q+2}\, F_{q,m}(\alpha);\Delta^2\right) \\
&- \frac{r}{m-2}\, G_{q+2,m-2}\left(\frac{q(m-2)}{m(q+2)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg] \\
&+ \Delta^2\Bigg[2G_{q+2,m}\left(\frac{q}{q+2}\, F_{q,m}(\alpha);\Delta^2\right) - G_{q+4,m}\left(\frac{q}{q+4}\, F_{q+4,m(\alpha);\Delta^2}\right) \\
&+ \frac{r}{m-2}\, G_{q+4,m-2}\left(\frac{q(m-2)}{m(q+4)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg]\Bigg\} \\
&+ \frac{\omega(1-\omega)^2}{N}\Bigg\{q\Bigg[G_{q+2,m}\left(\frac{q}{q+2}\, F_{q,m}(\alpha);\Delta^2\right) \\
&+ \frac{r}{m-2}\, G_{q+2,m-2}\left(\frac{q(m-2)}{m(q+2)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg] \\
&+ \Delta^2\Bigg[G_{q+4,m}\left(\frac{q}{q+4}\, F_{q+4,m}(\alpha);\Delta^2\right) \\
&+ \frac{r}{m-2}\, G_{q+4,m-2}\left(\frac{q(m-2)}{m(q+4)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg]\Bigg\}, \qquad (3.4)
\end{aligned}
$$

where $G_{r,s}(.;\Delta^2)$ denotes the cdf of a non-central F-distribution with (r,s) degrees of freedom and non-centrality parameter Δ^2.

Comparison between $\hat{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{Y}}$ can be easily done by making orders between the risk functions. But for the comparison of $\hat{\boldsymbol{\theta}}^{PT}$ and the others, based on BLF, we have the following abstracted results.

Under quadratic loss function the estimator $\hat{\boldsymbol{\theta}}^{PT}$ is always superior to the unbiased estimator $\bar{\boldsymbol{Y}}$ whenever

$$
\begin{aligned}
0 \le \Delta^2 \le q&\Bigg\{G_{q+2,m}\Big(\frac{q}{q+2}\, F_{q,m}(\alpha);\Delta^2\Big) - \frac{r}{m-2}\, G_{q+2,m-2}\left(\frac{q(m-2)}{m(q+2)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg\} \\
&\times\Bigg[2G_{q+2,m}\left(\frac{q}{q+2}\, F_{q,m}(\alpha);\Delta^2\right) - G_{q+4,m}\left(\frac{q}{q+2}\, F_{q+4,m}(\alpha);\Delta^2\right) \\
&+ \frac{r}{m-2}\, G_{q+4,m-2}\left(\frac{q(m-2)}{m(q+4)}\, F_{q,m}(\alpha);\Delta^2\right)\Bigg]^{-1}. \qquad (3.5)
\end{aligned}
$$

Otherwise $\bar{\boldsymbol{Y}}$ is superior. This conclusion remains valid under BLF using Lemma 1.1 (i), by taking $\boldsymbol{h}_1 = \boldsymbol{0}$ and $\boldsymbol{h}_2 = I(T^2 \le \frac{q}{m}\, F_{q,m}(\alpha))(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{Y}})$. Now consider that under H_0 : $\boldsymbol{\theta} = \boldsymbol{B}\boldsymbol{\eta}$, because $\boldsymbol{C}'\boldsymbol{B} = \boldsymbol{0}$ we have $\Delta^2 = N\boldsymbol{\eta}'\boldsymbol{B}'\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{\Sigma}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{B}\boldsymbol{\eta} = 0$. Hence the relative efficiency of $\hat{\boldsymbol{\theta}}^{PT}$ compared to $\bar{\boldsymbol{Y}}$ and $\hat{\boldsymbol{\theta}}$ based on quadratic loss function, under H_0, respectively given by

$$\begin{aligned} E(\hat{\boldsymbol{\theta}}^{PT} : \bar{\boldsymbol{Y}}) &= \left[1 - \frac{q}{p}\, D_0\right]^{-1} \geq 1 \\ E(\hat{\boldsymbol{\theta}}^{PT} : \hat{\boldsymbol{\theta}}) &= \left[1 - \frac{q(N-p-1)}{p(N-q-1)}\right] E(\hat{\boldsymbol{\theta}}^{PT} : \bar{\boldsymbol{Y}}) \geq \left[1 - \frac{q(N-p-1)}{p(N-q-1)}\right] \end{aligned}$$

since for every α, $G_{q,m}\,(F_{q,m}(\alpha); 0) = 1 - \alpha$; where

$$\begin{aligned} D_{\Delta^2} &= G_{q+2,m}\left(\frac{q}{q+2}\, F_{q,m}(\alpha); \Delta^2\right) \\ &\quad - \frac{r}{m-2}\, G_{q+2,m-2}\left(\frac{q(m-2)}{m(q+2)}\, F_{q,m}(\alpha); \Delta^2\right). \end{aligned}$$

Thus, using Lemma 1.1 (i), under H_0,

$$\left[1 - \frac{q(N-p-1)}{p(N-q-1)}\right] \leq E(\hat{\boldsymbol{\theta}}^{PT} : \hat{\boldsymbol{\theta}}) \leq E(\hat{\boldsymbol{\theta}}^{PT} : \bar{\boldsymbol{Y}}). \tag{3.6}$$

The inequality in (3.6) becomes strict whenever $D_0 \leq \left(1 - \frac{r}{m-2}\right)$ (see Srivastava and Saleh, 2005.)

Also using part (ii) of Lemma 1.1 and equation (3.2), the estimator $\bar{\boldsymbol{Y}}$ stays minimax under BLF. We can also conclude that under H_0 the restricted and preliminary test estimators are minimax. We close this section by some graphical results on preliminary test estimator performance w.r.t. the level of significance and weight coefficient w.

The following important points can be inferred from the Figures 1.

1. As w increases the risk values decrease. In other words, based on the structure of BLF, it confirms that if the model fit is good then the risk values are decreased as a natural consequence.

2. For approximate value $\Delta = 1$, the superiority order of the PTE for different levels of significance changes. In fact, for larger values α we consider better performance up to $\Delta = 1$, and vice versa for $\Delta > 1$.

4 Numerical Analysis

In this section we proceed to a numerical computation of a real data to show the application of the method discussed in this study. In this regard, consider the set of $N = 25$ observations given in Anderson (2003) taken from Frets (1921) consisting of head length and breadth of first and second son in a family. The full data is given in Table 1. In this case we have $p = 4$. Assume that the prior knowledge is given by (2.7) for an empirical Bayesian study. For the purpose of restricted and preliminary test estimation strategies we also need a matrix $\boldsymbol{B}$.

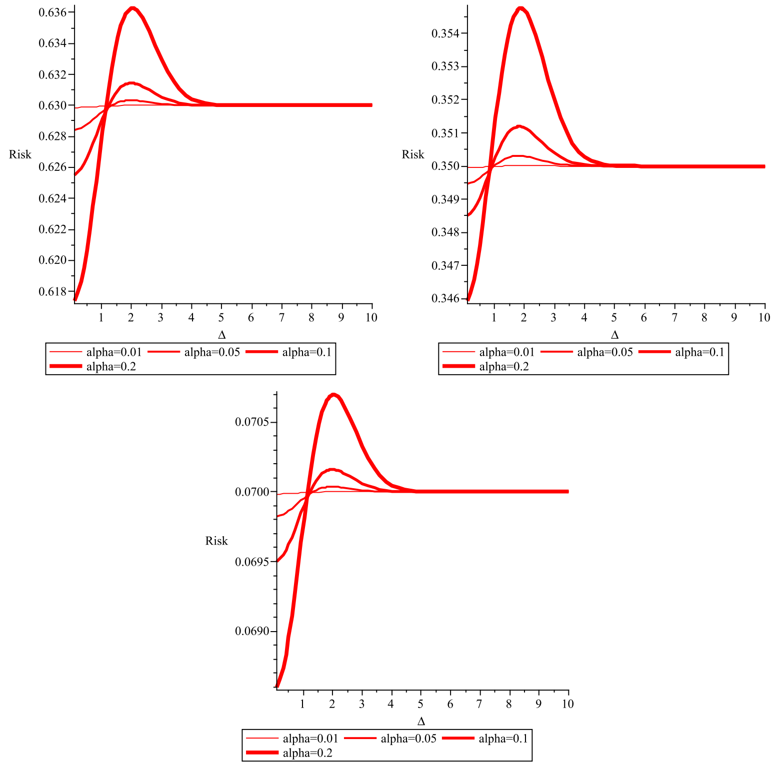

Figure 1: Risk Performance of PTE for $w = 0.1, 0.5, 0.9$

It is also important to find a matrix $\boldsymbol{C}$ for constructing the test statistic $\mathcal{L}_N$ such that $\boldsymbol{C}'\boldsymbol{B} = \mathbf{0}$. It may be noted that computationally simple methods to obtain a matrix $\boldsymbol{C}$ satisfying $\boldsymbol{C}'\boldsymbol{B} = \mathbf{0}$ are given in Srivastava (2002). With this in hand, suppose that one is desired to test the following null hypothesis

$$H_0 : \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \\ 100 \\ 500 \end{bmatrix} \quad \Rightarrow \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}, \boldsymbol{\eta} = \begin{bmatrix} 150 \\ 100 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 4 & 3 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

Obs	Y_1	Y_2	Y_3	Y_4	Obs	Y_1	Y_2	Y_3	Y_4
1	191	155	179	145	14	190	159	195	157
2	195	149	201	152	15	188	151	187	158
3	181	148	185	149	16	163	137	161	130
4	183	153	188	149	17	195	155	183	158
5	176	144	171	142	18	186	153	173	148
6	208	157	192	152	19	181	145	182	146
7	189	150	190	149	20	175	140	165	137
8	197	159	189	152	21	192	154	185	152
9	188	152	197	159	22	174	143	178	147
10	192	150	187	151	23	176	139	176	143
11	179	158	186	148	24	197	167	200	158
12	183	147	174	147	25	190	163	187	150
13	174	150	185	152					

Table 1: Head Lengths and Breadths of Brothers (Y_1=Head Length of First Son, Y_2=Head Breadth of First Son, Y_3=Head Length of Second Son, Y_4=Head Breadth of Second Son)

Also from the given data we get,

$$\bar{\boldsymbol{Y}} = \begin{bmatrix} 185.72 \\ 151.12 \\ 183.84 \\ 149.24 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 95.2933 & 52.8683 & 69.6617 & 46.1117 \\ 52.8683 & 54.3600 & 51.3117 & 35.0533 \\ 69.6617 & 51.3117 & 100.8067 & 56.5400 \\ 46.1117 & 35.0533 & 56.5400 & 45.0233 \end{bmatrix}.$$

Thus, it can be concluded that

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \begin{bmatrix} 4.86 \\ 40.85 \\ 22.86 \\ 55.45 \end{bmatrix}, \quad T^2 = 5073.41 \ \Rightarrow \\ \hat{\boldsymbol{\theta}}^{PT} &= \begin{bmatrix} 185.72 \\ 151.12 \\ 183.84 \\ 149.24 \end{bmatrix} - (1-\omega) I\left(9119.28 \leq \frac{2}{23}\, F_{2,23}(\alpha)\right) \begin{bmatrix} 180.85 \\ 110.26 \\ 160.97 \\ 93.78 \end{bmatrix}. \end{aligned}$$

At this stage any decision may be taken trough selecting a pre-specified level of significance. Because T^2 is large enough such that $I\left(9119.28 \leq \frac{2}{23}\, F_{2,23}(\alpha)\right) = 0$ for every reasonable α, therefore in overall we deduce that $\hat{\boldsymbol{\theta}}^{PT} = \bar{\boldsymbol{Y}}$. Overall, for all values α the null-hypothesis H_0 will be rejected. However, by making use of the equation (2.10) and the fact that $\mathcal{L}_N = 104868.20$, taking $g(\mathcal{L}_N) = 10^5\mathcal{L}_N^{-1}$ (resulting in Stein-type estimator), we can obtain better result than the latter (H_0 does not reject).

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OPTIMUM REGRESSION QUANTILES FOR THE INFERENCE OF THE PARAMETERS OF A SIMPLE REGRESSION MODEL WITH EXPONENTIAL ERRORS

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SUMMARY

Consider the simple linear regression model:$\mathrm{Y}_i = \beta_0 + \beta_1 x_i + \sigma z_i,\ (i = 1, \ldots, n)$, where $z_1, \ldots, z_n$ are i.i.d. errors with exponential distribution, $e^{-z},\ z \in R^+$. This paper deals with the estimation and tests of hypothesis regarding the parameters, $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)'$ based on a few "regression quantiles" introduced by Koenker and Bassett (1978). The question of optimum regression quantiles is addressed for the problems. Further, estimation of the conditional regression function is also considered along with the related optimum regression quantiles. In every case the optimum spacings are independent of the design matrix.

Keywords and phrases: ABLUE; Regression Quantiles; Optimum Spacings; JARE

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1 Introduction

It is well known that least squares estimators (LSE) of regression parameters are unbiased with minimum variance and the quadratic estimator of σ^2 is optimal in general. However, this may not be so for the model with exponential errors. For the maximum likelihood estimators (MLE) the Fisher information matrix for the parameter $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)'$ is given by the 3×3 matrix:

$$\frac{n}{\sigma^2}\begin{pmatrix} n & n\overline{x} & 1 \\ n\overline{x} & n(s^2 + \overline{x}^2) & \overline{x} \\ 1 & \overline{x} & 1 \end{pmatrix}.$$

In this paper we consider the estimation of $\boldsymbol{\theta}$ based on a few selected regression quantiles which is an extension of the sample quantiles in the location-scale model (See, Balakrishnan and Basu, 1995; David and Nagaraja, 2003; Harter, 1963; Sarhan and Greenberg, 1962; Saleh, 1981 and Saleh and Ali, 1966).

The objective of this paper is to basically, obtain (i) asymptotically best linear unbiased estimator (ABLUE) of $\boldsymbol{\theta}$ based on $k(3 \leq k \leq n)$ optimum regression quantiles, (ii) propose test-statistics for jointly testing $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)'$ under local alternatives and discuss the related optimum spacings and finally, (iii) propose ABLUE of a conditional quantile function, $y(\xi) = \beta_0 + \beta_1 x_0 + \sigma \ln(1-\xi)^{-1}, 0 < \xi < 1$ and related optimum spacings. Thus, as a first step, we assume that n is large and

$$(i) \lim_{n\to\infty} \bar{x}_n = \overline{x} \quad and \quad \lim_{n\to\infty} n^{-1} \begin{pmatrix} n & n\bar{x}_n \\ \bar{x}_n & \sum_{i=1}^{n} x_i^2 \end{pmatrix} = \begin{pmatrix} 1 & \overline{x} \\ \overline{x} & s^2 + \overline{x}^2 \end{pmatrix}.$$

Let $u = \ln(1-\lambda)^{-1}$ be the quantile function of the exponential distribution corresponding to the spacing λ $(0 < \lambda < 1)$ and let $q_0(\lambda) = 1 - \lambda$ be the corresponding density quantile function. Further, let $k(3 \leq k < n)$ be a fixed integer and consider the spacing vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)'$ satisfying the relation $0 < \lambda_1 < \cdots < \lambda_k < 1$.

Now, following Koenker and Bassett (1978) we obtain the k regression quantiles

$$\hat{\boldsymbol{\beta}}_{jn} = \left(\hat{\beta}_{jn}(\lambda_1), \ldots, \hat{\beta}_{jn}(\lambda_k)\right)', \; j = 0, 1$$

by minimizing $\sum_{j=1}^{n} \zeta_{\lambda_j}(y_j - \beta_0 - \beta_1 x_j)$, where $\zeta_\lambda(z) =\mid z \mid \{\lambda I(z > 0) + (1-\lambda) I(z < 0)\}$ with $I(A)$ as the indicator function of set A. Thus, using Theorem 4.2 of Koenker and Bassett (1978) we see that the $2k$–dimensional random variable

$$\left(\sqrt{n}[\hat{\boldsymbol{\beta}}_{0n}(\boldsymbol{\lambda}) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u}]', \sqrt{n}[\widehat{\boldsymbol{\beta}}_n(\boldsymbol{\lambda}) - \beta_1 \mathbf{1}_k]'\right)'$$

converges in law (as $n \to \infty$) to the $2k$–dimensional normal distribution with mean 0 and covariance matrix

$$\sigma^2 \begin{pmatrix} 1 & \overline{x} \\ \overline{x} & s^2 + \overline{x}^2 \end{pmatrix}^{-1} \otimes \boldsymbol{\Omega},$$

where $\boldsymbol{\Omega} = \left(\frac{\min(\lambda_i, \lambda_j) - \lambda_i \lambda_j}{(1-\lambda_i)(1-\lambda_j)}\right)$, and $\mathbf{1}_k = (1, 1, \ldots, 1)'$, a k–tuple of ones and $\mathbf{u} = (u_1, u_2, \ldots, u_k)'$, and $u_j = \ln(1-\lambda_j)^{-1}, \; j = 1, \ldots, k$.

These results will be used in the subsequent sections.

2 Joint Estimation of $(\beta_0, \beta_1, \sigma)'$

We obtain the ABLUE of $(\beta_0, \beta_1, \sigma)'$ by minimizing the quadratic form

$$[\hat{\boldsymbol{\beta}}_{0n}(\boldsymbol{\lambda}) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u}] \boldsymbol{\Omega}^{-1} [\hat{\boldsymbol{\beta}}_{0n}(\boldsymbol{\lambda}) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u}] + 2[\hat{\boldsymbol{\beta}}_{0n}(\boldsymbol{\lambda}) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u}]' \boldsymbol{\Omega}^{-1} [\hat{\boldsymbol{\beta}}_{1n}(\boldsymbol{\lambda}) - \beta_1 \mathbf{1}_k]$$
$$+ (s^2 + \overline{x}^2)[\widehat{\boldsymbol{\beta}}_{1n}(\boldsymbol{\lambda}) - \beta_1 \mathbf{1}_k]' \boldsymbol{\Omega}^{-1} [\widehat{\boldsymbol{\beta}}_{1n}(\boldsymbol{\lambda}) - \beta_1 \mathbf{1}_k]$$

with respect to $\beta_0,\ \beta_1$, and σ to obtain the normal equation

$$\mathbf{K}\boldsymbol{\theta}_n^* = \mathbf{V},$$

where

$$\mathbf{K} = \begin{pmatrix} K_1 & \overline{x}K_1 & K_3 \\ K_1\overline{x} & (s^2+\overline{x}^2)K_1 & \overline{x}K_3 \\ K_3 & \overline{x}K_3 & K_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\theta}_n^* = (\beta_{0n}^*, \beta_{1n}^*, \sigma_{0n}^*)' \quad \text{and} \quad \mathbf{V} = (V_0, V_1, V_2)',$$

with

$$\begin{aligned} &V_0 = Z_0 + \overline{x}Z_1, \qquad V_1 = \overline{x}V_1 + s^2 Z_1, \qquad V_2 = Z_0^* + \overline{x}Z_1^* \\ &Z_j = \mathbf{1}_k'\Omega^{-1}\widehat{\boldsymbol{\beta}}_{jn}, \qquad Z_j^* = \mathbf{u}_k'\Omega^{-1}\widehat{\boldsymbol{\beta}}_{jn}, \quad j = 0,1 \end{aligned}$$

and $\Delta = K_1K_2 - K_3^2$. The explicit form of K_1, K_2, and K_3 are given by

$$K_1 = 1/e^{u_1 - 1}, \quad K_2 = u_1^2/(e^{u_1-1}) + L, \ \text{ and } \ K_3 = u_1/(e^{u_1}-1),$$

where $L = \sum_{i=2}^{k} \frac{(u_i - u_{i-1})^2}{e^{u_i} - e^{u_{i-1}}}$ (see Saleh and Ali, 1966 and Saleh, 1981).

Now, as $n \to \infty$, the asymptotic distribution of

$$\left(\sqrt{n}(\beta_{0n}^* - \beta_0), \sqrt{n}(\beta_{1n}^* - \beta_1), \sqrt{n}(\sigma_n^* - \sigma)\right)',$$

follows the 3-dimensional normal distribution with mean $\mathbf{0}$ and dispersion matrix $\sigma^2\mathbf{K}^{-1}$, where $\mid \mathbf{K} \mid = s^2K_1\Delta$. Hence, the joint asymptotic relative efficiency ($JARE$) of $\boldsymbol{\theta}_n^*$ relative to the MLE, say $\overline{\boldsymbol{\theta}}_n$ is given by

$$JARE(\boldsymbol{\theta}_n^* : \overline{\boldsymbol{\theta}}_n) = \frac{K_1\Delta}{n(n-1)} = \frac{e^{-u_1}(e^{u_1}-1)^{-2}}{n(n-1)} Q_{k-1},$$

with $Q_{k-1} = \sum_{i=1}^{k-1} \frac{(t_i - t_{i-1})^2}{(e^{t_i} - e^{t_{i-1}})}$ as in Saleh and Ali (1966).

Notice that $JARE(\boldsymbol{\theta}_n^* : \overline{\boldsymbol{\theta}}_n)$ as a function of u_1 is decreasing and it has maximum near the origin given by

$$\lambda_1^0 = \frac{2}{2n+1} = 1 - e^{-u_1^0} \ \text{ giving } \ u_1^0 = \ln\left(\frac{2n+1}{2n-1}\right)$$

due to Saleh and Ali (1966). Then, conditionally on this spacing, the $JARE$ is given by

$$\frac{(2n-1)^3}{4n(n-1)(2n+1)} Q_{k-1}.$$

Thus, maximizing Q_{k-1} with respect to $(\lambda_1, \ldots, \lambda_k)$ one gets the optimum spacings given by Ogawa (1951), which yields the spacings given by

$$\lambda_1^* = \frac{1}{n+1/2} \quad \text{and} \quad \lambda_{j+1}^* = \frac{2+(2n-1)\lambda_j^0}{2n+1},\ j = 1, \ldots, k-1,$$

where λ_j^0 $(j = 1, \ldots, k-1)$ are the optimum spacings for the scale parameter alone which are available in Sarhan and Greenberg (1962).

Some tabular values of the $JARE$ are given in Table 1 below.

Table 1: Values of $JARE(\boldsymbol{\theta}_n^* : \overline{\boldsymbol{\theta}}_n)$ for selected $k = 2(1)10$ and $n = 50(10)100$

k	n	$JARE$	k	n	$JARE$	k	n	$JARE$	k	n	$JARE$
2	50	0.6349	4	70	0.8785	6	90	0.9371	9	50	0.9561
2	60	0.6370	4	80	0.8800	6	100	0.9382	9	60	0.9593
2	70	0.6385	4	90	0.8812	7	50	0.9416	9	70	0.9616
2	80	0.6396	4	100	0.8822	7	60	0.9448	9	80	0.9633
2	90	0.6405	5	50	0.9087	7	70	0.9470	9	90	0.9646
2	100	0.6412	5	60	0.9117	7	80	0.9487	9	100	0.9657
3	50	0.8041	5	70	0.9138	7	90	0.9500	10	50	0.9605
3	60	0.8068	5	80	0.9154	7	100	0.9510	10	60	0.9637
3	70	0.8087	5	90	0.9167	8	50	0.9502	10	70	0.9660
3	80	0.8101	5	100	0.9177	8	60	0.9533	10	80	0.9677
3	90	0.8112	6	50	0.9289	8	70	0.9556	10	90	0.9690
3	100	0.8121	6	60	0.9320	8	80	0.9573	10	100	0.9701
4	50	0.8735	6	70	0.9342	8	90	0.9586			
4	60	0.8764	6	80	0.9358	8	100	0.9596			

3 Test of Hypothesis on $(\beta_0, \beta_1, \sigma)'$

In this section, we consider the joint test of hypothesis:

$$H_0 : (\beta_0, \beta_1, \sigma)' = (\beta_0^0, \beta_1^0, \sigma^0)'$$

against

$$H_A : (\beta_0, \beta_1, \sigma)' \neq (\beta_0^0, \beta_1^0, \sigma^0)'$$

based on $\widehat{\boldsymbol{\beta}}_{jn} = (\widehat{\boldsymbol{\beta}}_{jn}(\lambda_1), \ldots, \widehat{\boldsymbol{\beta}}_{jn}(\lambda_k))',\ (j = 0, 1)$, where $(\beta_0^0, \beta_1^0, \sigma^0)'$ is a specified vector. In this context, our objective is to assess the asymptotic relative efficiency (ARE) of a test based on $(\beta^*_{0n}, \beta^*_{1n}, \sigma^*_n)'$ relative to a test based on $(\overline{\beta}_{0n}, \overline{\beta}_{1n}, \overline{\sigma}_n)'$. It is shown that the optimum spacings for this problem remains the same as in the estimation problem.

We now define the test statistics Q^*_n for testing H_0 against H_A as follows:

$$\begin{aligned}Q^*_n &= n(\sigma^0)^{-2}[K_1(\beta^*_{0n} - \beta^0_0)^2 + K_2(\sigma^*_n - \sigma^0)^2 + (s^2 + \overline{x}^2)K_1(\beta^*_{1n} - \beta^0_1)^2 \\ &\quad + 2\overline{x}K_3(\beta^*_{0n} - \beta^0_0)(\beta^*_{1n} - \beta^0_1) + 2\overline{x}K_3(\beta^*_{1n} - \beta^0_1)(\sigma^*_n - \sigma^0) + 2K_3(\beta^*_{0n} - \beta^0_0)(\sigma^*_n - \sigma^0).\end{aligned}$$

Then, the test function is defined by

$$\phi(Q^*_n) = \begin{cases} 1 & \text{if } \ Q^*_n \geq Q^*_{n,\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

Now under H_0, Q^*_n follows a central chi-squared distribution with 3 degrees of freedom (DF), and we take $Q^*_{n,\alpha} = \chi^2_{3,\alpha}$, which is the upper α%-tile of the chi-squared distribution. Similarly, we consider test-statistics based on $\overline{\boldsymbol{\theta}} = (\overline{\beta}_{0n}, \overline{\beta}_{1n}, \overline{\sigma}_n)'$ is given by

$$\begin{aligned}\overline{Q}_n &= n(\sigma^0)^{-2}[n(\overline{\beta}_{0n} - \beta^0_0)^2 + (\overline{\sigma}_n - \sigma^0)^2 + (s^2 + \overline{x}^2)(\overline{\beta}_{1n} - \beta^0_1)^2 \\ &\quad + 2\overline{x}(\overline{\beta}_{0n} - \beta^0_0)(\overline{\beta}_{1n} - \beta^0_1) + 2\overline{x}(\overline{\beta}_{1n} - \beta^0_1)(\overline{\sigma}_n - \sigma^0) + 2(\overline{\beta}_{0n} - \beta^0_0)(\overline{\sigma}_n - \sigma^0)\end{aligned}$$

giving the test-function

$$\phi(\overline{Q}_n) = \begin{cases} 1 & \text{if } \ \overline{Q}_n \geq \overline{Q}_{n,\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

As in the case Q^*_n, $\overline{Q}_n$ follows a central chi-squared distribution with 3 DF under H_0 and $\overline{Q}_{n,\alpha} = \chi^2_{3,\alpha}$ as before.

To find the asymptotic distribution of $Q^*_n(\overline{Q}_n)$ under H_A, we consider a sequence of local alternatives $\{A_n\}$, where

$$A_n : \beta_{0(n)} + \beta^0_0 + n^{-\frac{1}{2}}\delta_0, \qquad B_{1(n)} = \beta^0_1 + n^{-\frac{1}{2}}\delta_1, \qquad \overline{Q}_{(n)} = \sigma^0 + n^{-\frac{1}{2}}\delta_2,$$

where $\boldsymbol{\delta}' = (\delta_0, \delta_1, \delta_2) \neq (0,0,0)$ is some fixed real vector in $R^2 \times R^+$. Using the asymptotic distribution of $(\beta^*_{0n}, \beta^*_{1n}, \sigma^*_n)'$ and $(\overline{\beta}_{0n}, \overline{\beta}_{1n}, \overline{\sigma}_n)'$ under $\{A_n\}$, we find the asymptotic distribution of Q^*_n and $\overline{Q}_n$ follows the non-central chi-squared distribution with 3 degrees of freedom and the non-central parameters

$$\Delta^* = \boldsymbol{\delta}'\mathbf{K}\boldsymbol{\delta}/(\sigma^0)^2 \qquad \text{and} \qquad \overline{\Delta} = \boldsymbol{\delta}'\mathbf{I}\boldsymbol{\delta}/(\sigma^0)^2,$$

respectively. To compare Q^*_n and $\overline{Q}_n$, we note that the classical Pitman *ARE* result is applicable since the tests have same size α and similar non-central chi-squared distribution. So using Puri and Sen (1971) we obtain

$$ARE[Q^*_n : \overline{Q}_n] = \frac{\boldsymbol{\delta}'\mathbf{K}\boldsymbol{\delta}}{\boldsymbol{\delta}'\mathbf{I}\boldsymbol{\delta}}.$$

By Courant-Fisher theorem (Rao, 1973) the extremes of the ratio of two quadratic forms in $\boldsymbol{\delta}$ are given by

$$Ch_{\min}(\mathbf{KI}^{-1}) \leq \frac{\boldsymbol{\delta}'\mathbf{K}\boldsymbol{\delta}}{\boldsymbol{\delta}'\mathbf{I}\boldsymbol{\delta}} \leq Ch_{\max}(\mathbf{KI}^{-1}),$$

where $Ch_{\min}(A)$ and $Ch_{\max}(A)$ are minimum and maximum characteristic roots of A. In this case,

$$\mathbf{KI}^{-1} = \begin{pmatrix} \frac{(K_1-K_3)}{(n-1)} & \overline{x}[\frac{(K_1-K_3)}{(n-1)} - \frac{K_1}{n}] & \frac{(K_3-K_1)}{(n-1)} \\ 0 & \frac{K_1}{n} & 0 \\ \frac{[nK_3-K_1]}{(n-1)} & \overline{x}[\frac{nK_3-K_1}{(n-1)}] & \frac{(nK_2-K_3)}{(n-1)} \end{pmatrix}.$$

Further,

$$\begin{aligned} e_{(1)} + e_{(2)} + e_{(3)} &= tr(\mathbf{KI}^{-1}) \\ &= (n^2K_2 - 2K_3 + (2n-1)K_1) \\ &= \frac{(e^{u_1}-1)^{-1}}{n(n-1)}((nu_1-1)^2 + 2(n-1) + n^2(1-e^{-u_1})Q_{k-1}), \end{aligned}$$

where $e_{(i)},\ i = 1, 2, 3$ are the eigen values of the the matrix $\{\mathbf{KI}^{-1}\}$. This is a decreasing function of u_1. Hence, the optimal spacing for the maximum with respect to u_1 yields $2/(2n+1)$ as in the case of estimation. Rest of the spacings are obtained by maximizing Q_{k-1}.

Also, we have the product

$$e_{(1)}e_{(2)}e_{(3)} = \mid \mathbf{KI}^{-1} \mid = K_1\Delta/(n(n-1)),$$

which is same as the $JARE$ expression. Thus the optimum spacings are the same as the estimation problem. As a measure of the ARE of the test one may use $tr(\mathbf{KI}^{-1})/3$ or the geometric mean i.e. $(K_1\Delta)^{(1/3)}$.

4 Estimation of Conditional Regression Quantiles

Consider the conditional regression quantiles

$$Q(\xi) = \beta_0 + \beta_1 x_0 + \sigma \ln(1-\xi)^{-1}, \quad 0 < \xi < 1,$$

where x_0 and ξ are specified. We can estimate $Q(\xi)$ using the two estimators of $(\beta_0, \beta_1, \sigma)$, namely $(\beta^*_{0n}, \beta^*_{1n}, \sigma^*_n)'$ and $(\overline{\beta}_{0n}, \overline{\beta}_{1n}, \overline{\sigma}_n)'$ yielding

$$Q^*_n(\xi) = \beta^*_{0n} + \beta^*_{1n}x_0 + \sigma^*_n \ln(1-\xi)^{-1} \text{ and } \overline{Q}_n(\xi) = \overline{\beta}_{0n} + \overline{\beta}_{1n}x_0 + \overline{\sigma}_n \ln(1-\xi)^{-1}$$

with the respective asymptotic variance given by

$$Var[Q^*_n(\xi)] = \frac{\sigma^2}{n}\mathbf{L}'\mathbf{K}^{-1}\mathbf{L} \text{ and } Var[\overline{Q}_n(\xi)] = \frac{\sigma^2}{n}\mathbf{L}'\mathbf{I}^{-1}\mathbf{l},$$

where $\mathbf{L} = (1, x_0 \ln(1-\xi)^{-1})'$.

The ARE of $Q_n^*(\xi)$ relative to $\overline{Q}_n(\xi)$ is then given by

$$ARE[Q_n^*(\xi) : \overline{Q}_n(\xi)] = \frac{\mathbf{L}'\mathbf{I}^{-1}\mathbf{L}}{\mathbf{L}'\mathbf{K}^{-1}\mathbf{L}},$$

where

$$Ch_{\min}(\mathbf{KI}^{-1}) \leq ARE[Q_n^*(\xi) : \overline{Q}_n(\xi)] \leq Ch_{\max}(\mathbf{KI}^{-1}).$$

Thus, the optimum spacings of the k regression quantiles are the same as the spacings for the estimation and testing problems.

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