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ON NORMAL CONVERGENCE CRITERIA FOR SUMS OF ROW-WISE INDEPENDENT RANDOM VARIABLES

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SUMMARY

In this article we present new criteria for the asymptotic normality of sums of row-wise independent random variables. Besides providing an alternate approach for demonstrating a sum of independent random variables is asymptotically normal, our criteria provide new insight into the nature of asymptotic normality and the Lindeberg condition.

Keywords and phrases: Central limit theorem, infinitesimal array, Lindeberg condition

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1 Introduction

Classic criteria ensuring the asymptotic normality of sums of row-wise independent random variables (RV's) were determined early in the 20th century; see Le Cam (1986) for historical details. More recently, alternative criteria have been developed in terms of the first four moments, Kruglov (2004), and Lyapunov's condition, Kruglov (2008). Here we present new criteria for asymptotic normality based on Lévy's splitting of a RV X according to the size of its absolute value, Le Cam (1986): for any $\epsilon > 0$,

$$X = XI(|X| < \epsilon) + XI(|X| \geq \epsilon) \tag{1.1}$$

where $I(\cdot)$ is the indicator function $I(A) = 1$, A true; 0 otherwise. In addition to providing a new means of establishing the asymptotic normality of a sum of row-wise independent RV's, our criteria provide an alternate interpretation of asymptotic normality and, in particular, the Lindeberg condition.

2 Normal Convergence Criteria

Let X_{nj}, $j = 1, \ldots, k_n$, $\lim_n k_n = \infty$, be an array of row-wise independent RV's which is infinitesimal, i.e., for any $\epsilon > 0$ $\lim_n \max_{1 \le j \le k_n} P(|X_{nj}| \ge \epsilon) = 0$. (Here and in the sequel all limits are for $n \to \infty$.) Classic normal convergence criteria for $Y_n = \sum_{j=1}^{k_n} X_{nj}$ are provided by Loève (1977):

Theorem 1 (Classic Normal Convergence Criteria). *Let X_{nj} be an infinitesimal array of row-wise independent RV's. $Y_n = \sum_{j=1}^{k_n} X_{nj}$ converges in distribution to a normal RV with finite mean α and variance σ^2 ($Y_n \to_d N(\alpha, \sigma^2)$) if and only if for every $\epsilon > 0$ and a $\tau > 0$,*

I. $\lim_n \sum_{j=1}^{k_n} P(|X_{nj}| \ge \epsilon) = 0$,

II. $\lim_n \alpha_n(\tau) = \alpha$ *where* $\alpha_n(\tau) = \sum_{j=1}^{k_n} \alpha_{nj}(\tau)$ *and* $\alpha_{nj}(\tau) = \mathrm{E}[X_{nj} I(|X_{nj}| < \tau)]$,

III. $\lim_n \sigma_n^2(\tau) = \sigma^2$ *where* $\sigma_n^2(\tau) = \sum_{j=1}^{k_n} \sigma_{nj}^2(\tau)$ *and* $\sigma_{nj}^2(\tau) = \mathrm{E}[X_{nj}^2 I(|X_{nj}| < \tau)] - \alpha_{nj}^2(\tau)$.

Further, under I, II and III hold for any τ' in $(0, \tau]$.

The following theorem provides our new criteria based on 1.1.

Theorem 2 (Alternate Normal Convergence Criteria). *Let X_{nj} be an infinitesimal array of row-wise independent RV's. $Y_n = \sum_{j=1}^{k_n} X_{nj} \to_d N(\alpha, \sigma^2)$ if and only if there is a $\tau > 0$ such that for all $\epsilon \in (0, \tau]$,*

A. $W_n(\epsilon) = \sum_{j=1}^{k_n} X_{nj} I(|X_n j| \ge \epsilon)$ *converges in probability to zero* ($W_n \to_p 0$),

B. $\lim_n \alpha_n(\epsilon) = \alpha$, *and*

C. $\lim_n \sigma_n^2(\epsilon) = \sigma^2$.

Proof. Suppose $Y_n \to_d N(\alpha, \sigma^2)$. By Theorem 1, there is a $\tau > 0$ such that B and C hold for all $\epsilon \in (0, \tau]$. To show that A holds, we note that for any $\delta > 0$

$$\begin{aligned} P(|W_n(\epsilon)| \le \delta) &\ge P(W_n(\epsilon) = 0) \\ &\ge P\left(\bigcap_{j=1}^{k_n} \{|X_{nj}| < \epsilon\}\right) \\ &= 1 - P\left(\bigcup_{j=1}^{k_n} \{|X_{nj}| \ge \epsilon\}\right) \\ &\ge 1 - \sum_{j=1}^{k_n} P(|X_{nj}| \ge \epsilon). \end{aligned}$$

Thus by I of Theorem 1, $\lim_n P(|W_n(\epsilon)| \le \delta) \ge 1 - \lim_n \sum_{j=1}^{k_n} P(|X_{nj}| \ge \epsilon) = 1$ for any $\epsilon > 0$.

Conversely, suppose that for some $\tau > 0$, A, B, and C hold for $\epsilon \in (0, \tau]$. By 1.1 $X_{nj} = V_{nj}(\epsilon) + W_{nj}(\epsilon)$, where $V_{nj}(\epsilon) = X_{nj} I(|X_{nj}| < \epsilon)$ and $W_{nj}(\epsilon) = X_{nj} I(|X_{nj}| \geq \epsilon)$, so that $Y_n = V_n(\epsilon) + W_n(\epsilon)$ where $V_n(\epsilon) = \sum_{j=1}^{k_n} V_{nj}(\epsilon)$ and $W_n(\epsilon) = \sum_{j=1}^{k_n} W_{nj}(\epsilon)$. Henceforth we suppress ϵ in our notation. By A, $W_n \to_p 0$ for $\epsilon \in (0, \tau]$. Thus, by Slutsky's Theorem, $Y_n \to_d N(\alpha, \sigma^2)$ if $V_n \to_d N(\alpha, \sigma^2)$. Let $U_{nj} = (V_{nj} - \mathrm{E}[V_{nj}])/\sigma_n$ so that U_{nj} is a row-wise independent array, $\mathrm{E}[U_{nj}] = 0$, and $\sum_{j=1}^{k_n} \mathrm{Var}(U_{nj}) = 1$. By Lyapunov's Theorem, $\sum_{j=1}^{k_n} U_{nj} = U_n \to_d N(0, 1)$ if $\lim_n \sum_{j=1}^{k_n} \mathrm{E}[|U_{nj}|^3] = 0$. Since $|V_{nj}| < \epsilon$, $|V_{nj} - \mathrm{E}[V_{nj}]| \leq 2\epsilon$ so that

$$
\begin{aligned}
|V_{nj} - \mathrm{E}[V_{nj}]|^3 &= |V_{nj} - \mathrm{E}[V_{nj}]||V_{nj} - \mathrm{E}[V_{nj}]|^2 \\
&\leq 2\epsilon |V_{nj} - \mathrm{E}[V_{nj}]|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{j=1}^{k_n} \mathrm{E}[|U_{nj}|^3] &= \frac{1}{\sigma_n^3} \sum_{j=1}^{k_n} \mathrm{E}[|V_{nj} - \mathrm{E}[V_{nj}]|^3] \\
&\leq \frac{2\epsilon}{\sigma_n^3} \sum_{j=1}^{k_n} \mathrm{E}[(V_{nj} - \mathrm{E}[V_{nj}])^2] \\
&= \frac{2\epsilon}{\sigma_n^3} \sum_{j=1}^{k_n} \sigma_{nj}^2.
\end{aligned}
$$

Therefore $\lim_n \sum_{j=1}^{k_n} \mathrm{E}[|U_{nj}|^3] \leq 2\epsilon/\sigma$ by C. Since this inequality holds for all ϵ in $(0, \tau]$, $\lim_n \sum_{j=1}^{k_n} \mathrm{E}[|U_{nj}|^3] = 0$.

3 Remarks

If we interpret $X_{nj} I(|X_{nj}| < \epsilon)$ and $X_{nj} I(|X_{nj}| \geq \epsilon)$ as constituting the central and tail contributions, respectively, of X_{nj} then our normality criteria admit the following intuitive interpretation: A sum of asymptotically negligible, independent RV's $Y_n = \sum X_{nj}$ is asymptotically normal if and only if the X_{nj} are such that the sum of their tail contributions, W_n, is asymptotically negligible and the mean and variance of the sum of their central contributions, V_n, converge. Further, our criteria provide new insight into the Lindeberg condition, a construct admitting various characterizations; see, for example, Goldstein (2009). Suppose that $X_1, \ldots, X_n$ is a sequence of RV's with finite variances σ_j^2. Let $s_n^2 = \sum_{j=1}^{n} \sigma_j^2$. If the X_j satisfy the uniformly asymptotically negligible (UAN) condition $\lim_n \max_j \sigma_j^2/s_n^2 = 0$ then the normalized sum $Y_n = \sum_{j=1}^{n} X_j/s_n$ is asymptotically normal if and only if the Lindeberg condition is satisfied: for any $\epsilon > 0$

$$
\begin{aligned}
\lim_n (1/s_n^2) \sum_{j=1}^{n} \mathrm{E}[X_j^2 I(|X_j| \geq \epsilon s_n)] &= \lim_n \sum_{j=1}^{n} \mathrm{E}[(X_j/s_n)^2 I(|X_j/s_n| \geq \epsilon)] \\
&= 0.
\end{aligned}
$$

Since the X_j are UAN, the normalized summands $\tilde{X}_{nj} = X_j/s_n$ are infinitesimal. Thus, by Theorem 2 the Lindeberg condition is equivalent to the requirement that for every $\epsilon > 0$ $W_n = \sum_{j=1}^{n} \tilde{X}_{nj} I(|\tilde{X}_{nj}| \geq \epsilon) \to_p 0$.

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