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QUASI-EMPIRICAL BAYES MODELING OF MEASUREMENT ERROR MODELS AND R-ESTIMATION OF THE REGRESSION PARAMETERS

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SUMMARY

This paper deals with the R-estimation of the regression parameters of a measurement error model: $y_i = \beta_0 + \beta_1 x_i + e_i$ and $x_i^0 = x_i + u_i, i = 1, \ldots, n$. By combining the two sets of the information, an emaculate regression model is obtained using "quasi-empirical Bayes" estimates of the unknown covariates $x_1, \ldots, x_n$. The model produces consistent estimates of the attenuated slope and the intercept parameters and applies to broad range of regression problems. Asymptotic properties of the R-estimators are provided based on the "quasi-Bayes regression model". Some simulated results are presented as evidence of the performances of the estimators.

Keywords and phrases: ME model; asymptotic relative efficiency; rank estimators; quasi-empirical Bayes model; Theil-Sen estimator; asymptotic normality.

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1 Introduction

Consider the simple linear measurement errors (ME) model

$$\left.\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + e_i \\ x_i^0 &= x_i + u_i, \end{aligned}\right\} i = 1, \ldots, n, \tag{1.1}$$

where e_i is the response error and u_i is the measurement error in the regressor variable x_i, which is unobservable while x_i^0 is the corresponding observed value. Our problem is the R-estimation of the intercept and slope parameters $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ in the model (1.1). The commonly used methods of estimating $(\beta_0, \beta_1)^T$ are the least squares or maximum likelihood methods. For robust methods, one follows the R-,L- and M-estimation methods. The literature is void of all these estimation methods for the measurement error models. Some attempts have been made by Zamar (1989) and Cheng and Tsai (1995) to study the M-estimators by robustifying the variance and covariance matrices following Huber's(1981) technique. Recently, Jurečková, Picek and Saleh (2008) initiated testing of hypothesis problems with rank statistics in measurement error models. In this paper we attempt to develop the theory of R-estimation for $(\beta_0, \beta_1)^T$ for the measurement error model (1.1) similar to the theoretical works of Hodges and Lehmann (1963), Adichie (1967), Jaeckel (1972), Saleh and Sen (1978, 1987), Sen (1968), and Jurečková and Sen (1996) among others. The least squares estimators for the measurement error model are recalled in Section 2, "quasi-empirical" Bayes regression model is proposed in Section 3 and the joint R-estimation of the slope and the intercept in Section 4. Asymptotic properties are discussed in Section 5 along with the expression for the joint asymptotic relative efficiency (JARE) of the LSE as well as R-estimators are given for the ME model.

2 Least Squares Estimation

First we consider the traditional regression model

$$\mathbf{y} = \beta_0 \mathbf{1}_n + \beta_1 \mathbf{x} + \mathbf{e} \tag{2.1}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^{'}$, $\mathbf{x} = (x_1, \ldots, x_n)^{'}$ and $\mathbf{e} = (e_1, \ldots, e_n)_{'}$. In this case it is well known that the LSE of $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ is given by

$$\boldsymbol{\beta}_n^{*(L)} = \begin{pmatrix} \beta_{0n}^{*(L)} \\ \beta_{1n}^{*(L)} \end{pmatrix} = \begin{pmatrix} \overline{y}_n - \beta_{1n}^{*(L)} \overline{x}_n \\ \frac{(\mathbf{x} - \overline{x}_n \mathbf{1}_n)^{'} (\mathbf{y} - \overline{y}_n \mathbf{1}_n)}{(\mathbf{x} - \overline{x}_n \mathbf{1}_n)' (\mathbf{x} - \overline{x}_n \mathbf{1}_n)} \end{pmatrix} \tag{2.2}$$

where $\overline{x}_n = n^{-1} \mathbf{1}_n^{'} \mathbf{x}$ and $\overline{y}_n = n^{-1} \mathbf{1}_n^{'} \mathbf{y}$. Under the following assumptions

(i)

$$\lim_{n \to \infty} \overline{x}_n = \mu_x \quad \text{and} \quad \lim_{n \to \infty} \max_{1 \leq i \leq n} \frac{(x_i - \overline{x}_n)^2}{(\mathbf{x} - \overline{x}_n \mathbf{1}_n)' (\mathbf{x} - \overline{x}_n \mathbf{1}_n)} \to 0 \tag{2.3}$$

(ii) the Fisher information, $I(f) = \int_{-\infty}^{\infty} \left\{ -\frac{f'(x)}{f(x)} \right\}^2 f(x) dx < \infty$.

Then, the asymptotic normality of $\sqrt{n}(\boldsymbol{\beta}_n^{*(L)} - \boldsymbol{\beta})$ follows easily as

$$\begin{pmatrix} \sqrt{n}(\beta_{0n}^{*(L)} - \beta_0) \\ \sqrt{n}(\beta_{1n}^{*(L)} - \beta_0) \end{pmatrix} \sim N_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma_e^2 \begin{pmatrix} 1 + \frac{\mu_x^2}{\sigma_x^2} & -\frac{\mu_x}{\sigma_x^2} \\ -\frac{\mu_x}{\sigma_x^2} & \frac{1}{\sigma_x^2} \end{pmatrix} \right) \tag{2.4}$$

where $S_{nxx} = \frac{1}{n} (\mathbf{x} - \overline{x}_n \mathbf{1}_n)^{'} (\mathbf{x} - \overline{x}_n \mathbf{1}_n) \to \sigma_x^2$ as $n \to \infty$.

Now, consider the <u>measurement error model</u> (1.1) given by

$$\mathbf{y} = [\mathbf{1}_n, \mathbf{x}] \boldsymbol{\beta} + \mathbf{e}, \quad \mathbf{x}^0 = \mathbf{x} + \mathbf{u} \tag{2.5}$$

which may be written as

$$\mathbf{Y} = \left[\mathbf{1}_n, \mathbf{x}^0\right] \boldsymbol{\beta} + \mathbf{w} \tag{2.6}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^{'}$, $\mathbf{x}^0 = \left(x_1^0, \ldots, x_n^0\right)^{'}$, $\mathbf{w} = \mathbf{e} - \beta_1 \mathbf{u}$ and $\mathbf{u} = (u_1, \ldots, u_n)^{'}$.

$$\left[\mathbf{1}_n, \mathbf{x}^0\right] = \begin{bmatrix} 1 & x_1^0 \\ \vdots & \vdots \\ 1 & x_n^0 \end{bmatrix}_{n\times 2} . \tag{2.7}$$

Following Fuller (1987), we minimize

$$\left(\mathbf{y} - \left[\mathbf{1}_n, \mathbf{x}^0\right] \boldsymbol{\beta}\right)^{'} \left(\mathbf{y} - \left[\mathbf{1}_n, \mathbf{x}^0\right] \boldsymbol{\beta}\right) \tag{2.8}$$

with respect to $\boldsymbol{\beta}$ to obtain the least squares estimator (LSE) of $\boldsymbol{\beta}$ as

$$\tilde{\nu}_n^{(L)} = \left(\begin{bmatrix} \mathbf{1}_n^{'} \\ \mathbf{x}^{0'} \end{bmatrix} \left[\mathbf{1}_n, \mathbf{x}^0\right] \right)^{-1} \begin{bmatrix} \mathbf{1}_n^{'} \\ \mathbf{x}^{0'} \end{bmatrix} \mathbf{y} = \begin{pmatrix} \tilde{\nu}_{0n}^{(L)} \\ \tilde{\nu}_{1n}^{(L)} \end{pmatrix}. \tag{2.9}$$

We observe that

$$\tilde{\nu}_n^{(L)} = \boldsymbol{\beta} + \begin{pmatrix} 1 + \frac{n\overline{x}_n^0}{S_{xx}+S_{uu}} & -\frac{n\overline{x}_n^0}{S_{xx}+S_{uu}} \\ \frac{n\overline{x}_n^0}{S_{xx}+S_{uu}} & \frac{n}{S_{xx}+S_{uu}} \end{pmatrix} \begin{pmatrix} \overline{e}_n - \beta_1 \overline{u}_n \\ \frac{1}{n} S_{x^0 e} - \beta_1 \frac{1}{n} S_{x^0 u} + \overline{x}_n^0 \overline{e}_n - \beta \overline{x}_n^0 \overline{u}_n \end{pmatrix} \tag{2.10}$$

where $S_{ab} = (\mathbf{a} - \overline{a}\mathbf{1}_n)^{'} \left(\mathbf{b} - \overline{b}\mathbf{1}_n\right)$, $\mathbf{a} = (a_1, \cdots, a_n)^{'}$ and $\mathbf{b} = (b_1, \cdots, b_n)^{'}$. We make the following assumptions to obtain the relevant properties of $\tilde{\nu}_n^{(L)}$.

(A1) $\mathbf{e}$ and $\mathbf{u}$ are stochastically independent vectors such that $E(\mathbf{e}, \mathbf{u}) = (\mathbf{0}, \mathbf{0})$.

(A2) $\lim_{n\to\infty} \overline{x}_n = \mu_x \in R^1$ and $\lim_{n\to\infty} \max_{1\le i\le n} \frac{(\mathbf{x}_i - \overline{x}_n)^2}{(\mathbf{x}-\overline{x}_n\mathbf{1}_n)^{'}(\mathbf{x}-\overline{x}_n\mathbf{1}_n)} = 0$

(A3) The observed vector $\mathbf{x}^0$ is the adjusted vector $\mathbf{x}$ by an error-vector $\mathbf{u}$ with a known distribution function, say G_u assumed to possess up to fourth moment and $\mathrm{plim}_{n\to\infty} S_{nuu} = \sigma_u^2 > 0$, where

$$S_{nuu} = \frac{1}{n} \left(\mathbf{u} - \overline{u}_n \mathbf{1}_n\right)^{'} \left(\mathbf{u} - \overline{u}_n \mathbf{1}_n\right), \quad \overline{u}_n = \frac{1}{n} \mathbf{u}^{'} \mathbf{1}_n$$

and $\mathrm{plim}\{\frac{1}{n}\sum_{i=1}^n (u_i - \bar{u}_n)^4\} = \mu_4 \le \infty$. We assume σ_u^2 is known as an identifiability condition for estimation.

(A4) $\mathbf{x}$ and $\mathbf{u}$ are asymptotically uncorrelated i.e.

$$\mathrm{plim}_{n\to\infty} S_{nxu} = \mathrm{plim}_{n\to\infty} \left\{ \frac{1}{n} \left(\mathbf{x} - \overline{x}_n \mathbf{1}_n\right)^{'} \left(\mathbf{u} - \overline{u} \mathbf{1}_n\right) \right\} = 0$$

Then,

$$\text{plim}_{n\to\infty} S_{nx^0x^0} = \text{plim}_{n\to\infty} \frac{1}{n} \left(\mathbf{x}_0 - \overline{x}_n^0 \mathbf{1}_n\right)' \left(\mathbf{x}^0 - \overline{x}_n^0 \mathbf{1}_n\right) = \sigma_x^2 + \sigma_u^2$$

also

$$\text{plim}_{n\to\infty} \frac{1}{n} \mathbf{x}^{0'} \mathbf{1}_n = \mu_x. \tag{2.11}$$

Using the above assumptions we have

$$\tilde{\nu}_n^{(L)} = \boldsymbol{\beta} - \beta_1 \frac{S_{nuu}}{S_{nxx} + S_{nuu}} \begin{pmatrix} -\mu_x \\ 1 \end{pmatrix} + O_p(1) \tag{2.12}$$

where $\text{plim}_{n\to\infty} \frac{S_{nxx}}{S_{nxx}+S_{nuu}} = \text{plim}_{n\to\infty} \kappa_n = \kappa_x = \frac{\sigma_x^2}{\sigma_x^2+\sigma_u^2}$. Observe that

$$\tilde{\nu}_n^{(L)} = \boldsymbol{\beta} - \beta_1 (1 - \kappa_x) \begin{pmatrix} -\mu_x \\ 1 \end{pmatrix} + O_p(1) \tag{2.13}$$

where κ_x is reliability ratio (RR). Hence,

$$\text{plim}_{n\to\infty} \tilde{\nu}_n^{(L)} = \boldsymbol{\beta} - \beta_1 (1 - \kappa_x) \begin{pmatrix} -\mu_x \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \beta_1 (1 - \kappa_x) \mu_x \\ \kappa_x \beta_1 \end{pmatrix} = \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix} = \boldsymbol{\nu} \tag{2.14}$$

and $\tilde{\nu}_n^{(L)}$ estimates $\boldsymbol{\nu}$ consistently but not $\boldsymbol{\beta}$. Hence, the consistent estimator of $\boldsymbol{\beta}$ follows by writing

$$\tilde{\beta}_{1n}^{(L)} = \kappa_n^{-1} \tilde{\nu}_n^{(L)} \text{ and } \tilde{\beta}_{0n}^{(L)} = \overline{y}_n - \tilde{\beta}_{1n} \overline{x}_n^0 \tag{2.15}$$

where a consistent estimator of κ_x is given by

$$\hat{\kappa}_x = \left[1 - S_{nuu} (S_{nxx} + S_{nuu})^{-1}\right], \tag{2.16}$$

if κ_x is <u>unknown</u>. As for the asymptotic normality of $\tilde{\beta}_{1n}^{(L)} = \begin{pmatrix} \tilde{\beta}_{0n}^{(L)} \\ \tilde{\beta}_{1n}^{(L)} \end{pmatrix}$, we have the following result from Fuller (1987) and Schneeweiss (1976)

$$\begin{pmatrix} \sqrt{n}\left(\hat{\beta}_{0n}^{(L)} - \beta_0\right) \\ \sqrt{n}\left(\hat{\beta}_{1n}^{(L)} - \beta_1\right) \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \left(\sigma_e^2 + \kappa_x \beta_1^2 \sigma_u^2\right) \begin{pmatrix} 1 + \mu_x^2 G & -\mu_x G \\ -\mu_x G & G \end{pmatrix} \right\} \tag{2.17}$$

as $n \to \infty$, where

$$G = \left[\frac{1}{\kappa_x \sigma_x^2} + \frac{\beta_1^2 E[u^4 - 2\sigma_u^4]}{(\sigma_e^2 + \kappa_x \beta_1^2 \sigma_u^2)\, \sigma_x^4} \right].$$

If there is no measurement error, we obtain the asymptotic distribution of the LSE of the standard model given by (2.4). Using (2.4) and (2.17) and remembering that $0 \le \kappa_x \le 1$, and $\Delta^2 = \left(\sigma_e^{-2} \kappa_x \beta_1^2 \sigma_u^2\right)$, the joint asymptotic relative efficiency (JARE) is given by:

$$JARE\left(\tilde{\boldsymbol{\beta}}_n^{(L)}; \boldsymbol{\beta}_n^{*(L)}\right) = \left[(1 + \Delta^2) G\right]^{-1}.$$

If there is no measurement error, then the JARE reduces to unity.

3 Quasi-Empirical Bayes Regression Models

In this section, we develop an emaculate regression model using the component information of the ME model. Let us consider the general case of the ME model under non-normal errors

$$\left.\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + e_i \\ x_i^0 &= x_i + u_i \end{aligned}\right\} \quad i = 1, \ldots, n \tag{3.1}$$

where $e_1, \ldots, e_n$ are i.i.d.r.v. with the symmetric pdf with finite Fisher's Information, $I(f_0)$ as follows:

$$f_e(e) = f_0(y - \beta_0 - \beta_1 x) \tag{3.2}$$

such that

$$E(y_i) = \beta_0 + \beta_1 x_i, \quad i = 1, \ldots, n \text{ and } Var(y_i) = \sigma_e^2. \tag{3.3}$$

We assume that

(i) $x_1^0, x_2^0, \ldots, x_n^0$ are independently distributed with the symmetric pdf

$$f_{\mathbf{x}^0}\left(x_i^0\right) = \frac{1}{\lambda_u} f_1\left(\frac{x_i^0 - x_i}{\lambda_u}\right) = \frac{1}{\lambda_u} f_1\left(\frac{u_i}{\lambda_u}\right) \tag{3.4}$$

with $E\left(x_i^0 | x_i\right) = x_i$ and the *known* variance, $\lambda_u^2 \sigma_u^2$.

(ii) The unknown covariates $x_1, \cdots, x_n$ are independently distributed with the symmetric pdf

$$f_{\mathrm{X}}(x_i) = \frac{1}{\lambda_x} f_2\left(\frac{x_i - \mu_x}{\lambda_x}\right)$$

with $E(x_i) = \mu_x$ and variance $\lambda_x^2 \sigma_x^2$.

(iii) $S_{nxu} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{x}_n\right)\left(u_i - \overline{u}_n\right) \xrightarrow{P} 0$

Further, we have

(a) $S_{nuu} = \frac{1}{n}\sum_{i=1}^{n}\left(u_i - \overline{u}_n\right)^2 \xrightarrow{P} \lambda_u^2\sigma_u^2$

(b) $S_{nxx} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{x}_n\right)^2 \rightarrow \lambda_x^2\sigma_x^2$ and

(c) $S_{nx^0x^0} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i^0 - \overline{x}_n^0\right)^2 \xrightarrow{P} \lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2 = \sigma_{x^0}^2$ (say)

Firstly, we note that the model (2.6) has the inherent problem that x_i^0 is <u>not uncorrelated</u> with $e_i - \beta_1 u_i$. As a result, the LSE $\tilde{\nu}_{1n}^{(L)}$ is not consistent for β_1, though it is consistent for the attenuated slope $\nu_1 = \kappa_x\beta_1$. Now we consider the model (where the <u>slope has to be determined</u>) as

$$y_i = \beta_0 + bx_i^0 + z_i \quad i = 1, \ldots, n, \tag{3.5}$$

so that for some b, we have

$$\operatorname{p}\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}\left(x_i^0 - \overline{x}_n^0\right)'\left(z_i - \overline{z}\right)\right\} = 0. \tag{3.6}$$

Accordingly, for some b, the LHS of (3.7) in the bracket may be written as:

$$(\beta_1 - b)\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x}_n)^2 - b\frac{1}{n}\sum(u_i - \bar{u}_n)^2 + b\frac{1}{n}\sum(x_i - \overline{x}_n)(u_i - \bar{u}_n) \tag{3.7}$$

$$+\frac{1}{n}\sum(x_i - \overline{x}_n)(e_i - \bar{e}_n) + \frac{1}{n}\sum(u_i - \overline{u}_n)(e_i - \bar{e}_n)\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x}_n)(u_i - \bar{u}_n)$$

Under the assumptions A3 - A4 and (3.5), as $n \to \infty$, it reduces to

$$(\beta_1 - b)\lambda_x^2\sigma_x^2 - b\lambda_u^2\sigma_u^2 = 0. \tag{3.8}$$

Solving for b, we obtain $b = \kappa_x\beta_1$, where $\kappa_x = \lambda_x^2\sigma_x^2/(\lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2)$. Thus, the emaculate regression model representing the ME model may be written as:

$$y_i = \nu_0 + \nu_1 x_i^0 + z_i \quad i = 1, \ldots, n, \tag{3.9}$$

with

$$\nu_0 = \beta_0 + \beta_1(1 - \kappa_x)\mu_x, \nu_1 = \kappa_x\beta_1 \text{ and } z_i = e_i + \beta_1\{(1 - \kappa_x)(x_i - \mu_x) - \kappa_x u_i\}.$$

Note that the model (3.11) may be obtained by the Bayesian approach, and is discussed by Whittamore (1989). For the general error distribution given by (3.2) and (3.5), we follow the quasi-empirical Bayes" method (see Saleh (2006), Ch. 4) for estimating the unknown

x_i's in (1.1). Following the method, we obtain the "Stein-type estimates" of the unknown covariates x_i's as

$$x_i^s = (n-3)\,\overline{x}_n^0 \mathcal{L}_n^{-1} + x_i^0 \left\{1 - (n-3)\,\mathcal{L}_n^{-1}\right\} \tag{3.10}$$

where $\mathcal{L}_n = nS_{nx^0x^0}/\lambda_u^2\sigma_u^2$ is the test-statistic for testing the null-hypothesis

$$H_0 : x_i = \mu_x \quad \forall \quad i \quad \text{vs} \quad H_1 : x_i \neq \mu_x \quad \text{for at least one } i \text{ under (3.5).} \tag{3.11}$$

We may re-write x_i^s as

$$x_i^s = (1-\hat{\kappa}_x)\,\overline{x}_n^0 + \hat{\kappa}_x x_i^0, i = 1, \ldots, n \tag{3.12}$$

where $\hat{\kappa}_x = \left\{1 - (n-3)\left[\lambda_u^2\sigma_u^2/nS_{nx^0x^0}\right]\right\}$ where $\hat{\kappa}_x$ is a consistent estimator of κ_x. Further, from the fact that

(i) $\overline{x}_n^0 \xrightarrow{P} \mu_x$, and (ii) $(1/n)S_{nx^0x^0} \xrightarrow{P} \lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2$, it follows that $\text{plim}\,\hat{\kappa}_x = \kappa_x = \lambda_x^2\sigma_x^2/(\lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2)$ (the reliability ratio) so that we may write the parametric version of (3.17) as

$$E\left(x_i|x_i^0\right) \equiv (1-\kappa_x)\,\mu_x + \kappa_x x_i^0, \quad i = 1, \ldots, n \tag{3.13}$$

The expression (3.18) allows us to write the "quasi - empirical Bayes regression model" corresponding to the ME model (1.1) as

$$y_i = \beta_0 + \beta_1\,(1-\kappa_x)\,\mu_x + \kappa_x\beta_1 x_i^0 + z_i \tag{3.14}$$

with $z_i = e_i + \beta_1\left\{(1-\kappa_x)\,(x_i-\mu_x) - \kappa_x u_i\right\}$, $(i = 1, \ldots, n)$ as in (3.11). This model is the conditional expectation of y_i given x_i^0 under normal errors. Writing

$$z_i = e_i - \beta_1\{\kappa_x u_i - (1-\kappa_x)(x_i-\mu_x)\} = e_i - v_i, \quad v_i = \nu_1 u_i - \beta_1(1-\kappa_x)(x_i-\mu_x), \quad \text{(say)},$$

we define c.d.f. and p.d.f. as a convolution of e_i and v_i as

$$F^*(x) = \int_{-\infty}^{\infty} F(x-\kappa_x\beta_1 v)h_\theta(v)dv \quad \text{and} \quad f^*(x) = \kappa_x\beta_1\int_{-\infty}^{\infty} f(x-\kappa_x\beta_1 v)h_\theta(v)dv \tag{3.15}$$

where $h_\theta(v)$ is the p.d.f. of the v's depending on $\theta = (\kappa_x, \beta_1)$ given by $v_i = \{u_i - (1-\kappa_x)\kappa_x^{-1}(x_i-\mu_x)\}$, $i = 1, \ldots, n$. Now assume $f_1 = f_2 = f_0$ in (3.5(i) & (ii)). Then,

$$I(f^*) = \int_{-\infty}^{\infty}\left\{-\frac{f^{*'}(x)}{f^*(x)}\right\}^2 f^*(x)dx \leq \kappa_x\beta_1 I(f). \tag{3.16}$$

Under this model with the associated assumptions, the LSE (2.15) of $\boldsymbol{\beta}$ has an asymptotic distribution given by

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_n^{(L)} - \boldsymbol{\beta}) \approx N_2\Big\{\mathbf{0}, (\sigma_e^2 + \kappa_x\lambda_u^2\beta_1^2\sigma_u^2)\boldsymbol{\Sigma}\Big\} \tag{3.17}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\mu_x^2 G^* & -\mu_x G^* \\ -\mu_x G^* & G^* \end{pmatrix} \quad \text{with } G* = \left[\frac{1}{\kappa_x \sigma_x^2} + \frac{\beta_1^2 E[u^4 - 2\sigma_u^4]}{(\sigma_e^2 + \kappa_x \lambda_u^2 \beta_1^2 \sigma_u^2)\,\sigma_x^4}\right].$$

Thus, the JARE($\tilde{\boldsymbol{\beta}}_n^{(L)} : \beta_n^{*(L)}$) $= [(1+\Delta^{*2})G*]^{-1}$ with $\Delta^{*2} = \sigma_e^{-2}\kappa_x \lambda_u^2 \beta_1^2 \sigma_u^2$. Further, define the Fisher scores

$$\varphi(u, f^*) = -\frac{f^{*'}(F^{*-1}(u))}{f^*(F^{*-1}(u))}, \quad 0 < u < 1 \quad \text{and} \quad \gamma^*(u, f^*) = \int_0^1 \varphi(u)\varphi(u, f^*)du.$$

based on the distribution f^* of the errors, z_i's.

In the next section we develop the R-estimators of $\boldsymbol{\beta}$.

4 R-estimators of Regression Parameters

Consider the traditional model (2.1) again with the associated assumptions in (2.3). We consider the R-estimators of $\boldsymbol{\beta} = (\beta_0, \beta_1)'$ using linear rank statistics as given by Saleh and Sen (1978)

$$L_n^0(b) = n^{-1/2} \sum_{i=1}^n (x_i - \bar{x}_n) a_n^{\varphi}(R_i(b))$$

and

$$T_n^0(a, b) = n^{-1/2} \sum_{i=1}^n a_n^{\varphi^+}(R_{ni}^+(a, b))(y_i - a - bx_i) \tag{4.1}$$

where $R_i(b)$ is the rank of $y_i - bx_i$ among $y_1 - bx_1, \ldots, y_n - bx_n$ and $R_i^+(b)$ is the rank of $|y_i - bx_i|$ among $|y_1 - bx_1|, \ldots, |y_n - bx_n|$. Now, consider the score $a_n^{\varphi}(i)\,(a_n^+(i))$ generated by a nondecreasing, square integrable score function $\varphi(\varphi^+) : (0,1) \to R^1$ in either of the two following ways:

$$a_n^{\varphi}(i) = \mathsf{E}\varphi(U_{(i)}) \quad \text{or} \quad \varphi\left(\frac{i}{n+1}\right), \quad i = 1, \ldots, n, \tag{4.2}$$

$$a_n^{\varphi^+}(i) = \mathsf{E}\varphi(U_{(i)}) \quad \text{or} \quad \varphi^+\left(\frac{i}{n+1}\right), \quad i = 1, \ldots, n,$$

where $U_{(1)} \leq \cdots \leq U_{(n)}$ the order statistics of a sample of size n from the uniform distribution $\mathsf{U}(0,1)$. Here $\varphi^+(u) = \varphi\left(\frac{1+u}{2}\right)$ with the condition: $\varphi(u) + \varphi(1-u)$. We define

$$A_{\varphi}^2 = \int_0^1 \varphi^2(u)du - \left(\int_0^1 \varphi(u)du\right)^2, \tag{4.3}$$

$$\gamma(\varphi, f) = \int_0^1 \varphi(u)\varphi(u, f)du, \quad \varphi(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}$$

with $0 < u < 1$. Following Saleh and Sen (1978) we define the R-estimators of β_1 as

$$\beta_{1n}^{*(R)} = \frac{1}{2}\{\sup[b : L_n(b) > 0] + \inf[b : L_n(b) < 0]\} \tag{4.4}$$

and that of β_0 by

$$\beta_{0n}^{*(R)} = \frac{1}{2}\left\{\sup[a : T(a, \beta_{1n}^{*(R)}) > 0] + \inf[a : T(a, \beta_{1n}^{*(R)}) > 0]\right\} \tag{4.5}$$

where

$$T_n^0\left(a, \beta_{1n}^{*(R)}\right) = n^{1/2}\sum_{i=1}^{n} a_n^{\varphi^+}\left(R_{ni}(a, \beta_{1n}^{*(R)})\right)\mathsf{sgn}(y_i - a - \beta_{1n}^{*(R)}x_i) \tag{4.6}$$

is the aligned rank statistics. Now, using asymptotic linearity results of Jurečková (1971) given by

$$\sup_{|\delta|<c}\left\{n^{1/2}\left|L_n^0(n^{-1/2}\delta) - L_n^0(0) + \gamma(\varphi, f)\sigma_x^2\delta\right|\right\} \to O_p(1) \tag{4.7}$$

and

$$\sup_{|\delta_i|<c_i,\, i=1,2}\left\{n^{1/2}\left|T_n^0(n^{-1/2}\delta_1, n^{-1/2}\delta_2) - T_n^0(0,0) + \gamma(\varphi, f)[\delta_1 + \delta_2\mu_x]\right|\right\} = O_p(1). \tag{4.8}$$

From the non-decreasing property of $L_n(b)$ and the linearity results given by (4.7) and (4.8) we obtain the fact that $n^{1/2}|\beta_{0n}^{*(R)} - \beta_0| = O_p(1)$ and $n^{1/2}|\beta_{1n}^{*(R)} - \beta_1| = O_p(1)$ and the asymptotic normality given by Saleh (2006) as

$$\begin{pmatrix}\sqrt{n}\left(\beta_{0n}^{*(R)} - \beta_0\right)\\ \sqrt{n}\left(\beta_{1n}^{*(R)} - \beta_1\right)\end{pmatrix} \sim \mathsf{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \frac{A_\varphi^2}{\gamma^2(\varphi, f)}\begin{pmatrix}1 + \frac{\mu_x^2}{\sigma_x^2} & -\frac{\mu_x}{\sigma_x^2}\\ -\frac{\mu_x}{\sigma_x^2} & \frac{1}{\sigma_x^2}\end{pmatrix}\right), \tag{4.9}$$

Thus, ARE of $\beta_n^{*(R)}$ compared to $\beta_n^{(L)}$ is given by

$$\mathsf{ARE}\left(\beta_n^{*(R)}, \beta_n^{(L)}\right) = \sigma_e^2\frac{\gamma^2(\varphi, f)}{A_\varphi^2} \tag{4.10}$$

If f is normal and we have used the Wilcoxon's score, $\varphi(u) = 2u - 1$, then

$$\mathsf{ARE}\left(\beta_n^{*(R)}, \beta_n^{(L)}\right) = \frac{3}{\pi}. \tag{4.11}$$

Now, we consider the <u>measurement error model</u> (3.1) together with the "quasi - empirical Bayes" model (3.19) along with the assumptions and (3.5). Let $\varphi : (0,1) \to \mathbb{R}^1$ be the score functions belonging to the class of non-constant, nondecreasing and square integrable functions. Let us consider the scores $a_n^\varphi(1), \ldots, a_n^\varphi(n)$ generated by score functions $\varphi : (0,1) \to \mathbb{R}$ as defined by (4.2). Define the linear rank statistic (LRS) for the slope parameter of the model (3.19) based on the observations $\{(x_i^0, y_i)|i = 1, \ldots, n\}$ as

$$L_n(b) = n^{-1/2}\sum_{i=1}^{n}(x_i^0 - \bar{x}_n^0)a_n^\varphi(R_i(b)), \tag{4.12}$$

where $R_i(b)$ is the rank of $y_i - bx_i^0$ among $y_1 - bx_1^0, \ldots, y_n - bx_n^0$. Notice that $y_1 - bx_1^0, \ldots, y_n - bx_n^0$ are independently distributed random variables with pdf $f*$. Hence,

$$\mathcal{P}\left\{R_1(b), \ldots, R_n(b)\right\} = \frac{1}{n!} \tag{4.13}$$

Thus (4.12) is a legitimate linear rank statistics for the estimation of the slope parameter of the model. Now, setting $L_n(b) = 0$, we define the solution as the estimator of $\nu_1 = \kappa_x \beta_1$ which is the slope of the "quasi - empirical Bayes" model. Due to the non-decreasing nature of $L_n(b)$ as a function of b, we define R-estimator $\hat{\nu}_{1n}^{(R)}$ routinely as

$$\hat{\nu}_{1n}^{(R)} = \frac{1}{2}\left\{\sup\left[b; L_n(b) > 0\right] + \inf\left[b; L_n(b) > 0\right]\right\} \tag{4.14}$$

which satisfies the asymptotic linearity results of Jurečková (1971) given by

$$\sup_{|\delta^*|<c}\left\{\left|L_n(\nu_1 + n^{-1/2}\delta^*) - L_n(\nu_1) + \gamma(\varphi, f^*)\sigma_{x^0}^2\delta^*\right| \delta^* \in R'\right\} = O_p(1) \tag{4.15}$$

where $\delta^* = \kappa_x \delta$ with $\kappa_x = \lambda_x^2\sigma_x^2/(\lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2)$. To justify its consideration we look at the "quasi- empirical Bayes" model in vector form as:

$$\mathbf{y} = \left[\beta_o + \beta_1(1-\kappa_x)\mu_x\right]\mathbf{1}_n + \kappa_x\beta_1\mathbf{x}^0 + \mathbf{z}$$

where $\mathbf{z}$ is the error vector of independent components, so that the R-estimator defined by (4.14) estimates $\nu_1 = \kappa_x\beta_1$. The proof of (4.15) may be found in Jurečková and Sen (1996), Hajek et al (1999) and Koul (2000). Since $L_n(b)$ is non-increasing function of b and (4.15) hold we conclude that $n^{1/2}\left|\hat{\nu}_{1n}^{(R)} - \nu_1\right| = O_p(1)$ where $\nu_1 = \kappa_x\beta_1$. Hence, $\hat{\nu}_{1n}^{(R)}$ is a consistent estimator of ν_1. Therefore, $\kappa_x^{-1}\hat{\nu}_{1n}^{(R)}$ is also a consistent estimator of β_1. Hence, inserting $\delta^* = n^{1/2}\left(\hat{\nu}_{1n}^{(R)} - \nu_1\right)$ in (4.15) we obtain

$$n^{1/2}(\hat{\nu}_{1n}^{(R)} - \nu_1) = \frac{1}{\gamma(\varphi, f^*)}L_n^*(\gamma_1) + o_p(1), \tag{4.16}$$

where

$$L_n^*(\nu_1) = n^{-1/2}\sum_{i=1}^{n}(x_i^0 - \bar{x}_n^0)\varphi(F^*(z_i)) + o_p(1) \approx N(0, A_\varphi^2\sigma_{x^0}^2), \tag{4.17}$$

where $\sigma_{x^0}^2 = \lambda_x^2\sigma_x^2 + \lambda_u^2\sigma_u^2$ Thus, we obtain

$$n^{1/2}(\hat{\nu}_{1n}^{(R)} - \nu_1) \sim \mathsf{N}\left(0, \frac{A_\varphi^2}{\gamma^2(\varphi, f^*)\sigma_{x^0}^2}\right). \tag{4.18}$$

Consequently,

$$n^{1/2}(\hat{\beta}_{1n}^{(R)} - \beta_1) \sim \mathsf{N}\left(0, \frac{A_\varphi^2}{\gamma^2(\varphi, f^*)\kappa_x\sigma_x^2}\right). \tag{4.19}$$

The asymptotic relative efficiency (ARE) of $\hat{\beta}_{1n}^{(R)}$ relative to $\hat{\beta}_{1n}^{(L)}$ is given by

$$\mathsf{ARE}(\hat{\beta}_{1n}^{(R)} : \hat{\beta}_{1n}^{(L)}) = \left(\sigma_e^2 + \kappa_x \lambda_u^2 \beta_1^2 \sigma_u^2\right) \frac{\gamma^2(\varphi, f^*)}{A_\varphi^2} \left[1 + \frac{\kappa_x \beta_1^2 E[u^4 - 2\sigma_u^4]}{(\sigma_e^2 + \kappa_x \lambda_u^2 \sigma_u^2 \beta_1^2)\lambda_x^2 \sigma_x^2}\right]. \tag{4.20}$$

If $\varphi(u) = 2u - 1$ i.e. Wilcoxon's score, the ARE turns out to be

$$48 \left(\sigma_e^2 + \kappa_x \lambda_u^2 \beta_1^2 \sigma_u^2\right) \left[1 + \frac{\kappa_x \beta_1^2 E[u^4 - 2\sigma_u^4]}{(\sigma_e^2 + \kappa_x \lambda_u^2 \sigma_u^2 \beta_1^2)\lambda_x^2 \sigma_x^2}\right] B^2(F^*) \tag{4.21}$$

and, the ARE of $\hat{\beta}_{1n}^{(R)}$ w.r.t $\hat{\beta}_{1n}^{*(R)}$ is simply given by

$$\kappa_x \frac{\gamma^2(\varphi, f^*)}{\gamma^2(\varphi, f)} \le \kappa_x \frac{B^2(F^*)}{B^2(F)} \le \int h_\theta^2(t) dt \tag{4.22}$$

where

$$B^2(F) = \left(\int f^2(x) dx\right)^2 \text{ and } B^2(F^*) = \left(\int_{-\infty}^{\infty} {f^*}^2(x) dx\right)^2.$$

The above result follows since $\varphi(u) = 2u - 1$ and

$$\gamma(\varphi, f) = \int \varphi(F(x)) f^2(x) dx = 2 \int f^2(x) dx \tag{4.23}$$

$$\gamma^*(\varphi, f^*) = 2 \int {f^*}^2(x) dx \tag{4.24}$$

where

$$f^*(x) = \int f(x - t) h_\theta(t) dt, \quad \frac{\gamma^*(\varphi, f^*)}{\gamma(\varphi, f)} = \frac{\int (\int f(x-t) h_\theta(t) dt)^2 dx}{\int f^2(x) dx} \le \int h_\theta^2(t) dt.$$

5 Joint R-estimation of Intercept and Slope

In this section, we consider the estimation of the intercept and slope parameters jointly by using the two linear rank statistics as in Saleh and Sen (1978):

$$L_n(b) = n^{-1/2} \sum_{i=1}^{n} (x_i^0 - \bar{x}_n^0) a_n^\varphi(R_i(b)) \tag{5.1}$$

and

$$T_n(a, b) = n^{-1} \sum_{i=1}^{n} a_n^{\varphi^+}(R_i^+(a, b)) \mathsf{sgn}(y_i - a - b x_i^0), \tag{5.2}$$

where $R_i^+(a, b)$ is the rank of $|y_i - a - b x_i^0|$ among $|y_1 - a - b x_1^0|, \ldots, |y_n - a - b x_n^0|$ with the scores $a_n^{\varphi^+}(1), \ldots, a_n^{\varphi^+}(n)$ defined by

$$a_n^{\varphi^+}(i) = \mathsf{E}\varphi^+(U_{(i)}) \quad \text{or} \quad \varphi^+\left(\frac{i}{n+1}\right) \tag{5.3}$$

with

$$\varphi^{+}(u)=\varphi\left(\frac{1+u}{2}\right),\quad \varphi(u)+\varphi(1-u)=0,\quad 0<u<1. \tag{5.4}$$

Let $\hat{\beta}_{1n}^{(R)}$ be the R-estimator of the slope parameter using $\hat{\nu}_{1n}^{(R)}=\kappa_x\beta_1$. Then to obtain the R-estimator of the intercept parameter, we consider the aligned statistics

$$T_n(a,\hat{\nu}_{1n}^{(R)})=n^{-1/2}\sum_{i=1}^{n}a_n^{\varphi^{+}}(R_i(a,\hat{\nu}_{1n}^{(R)}))\mathsf{sgn}(y_i-a-\hat{\nu}_{1n}^{(R)}x_i^0). \tag{5.5}$$

This allows us to obtain the R-estimator as

$$\hat{\nu}_{0n}^{(R)}=\frac{1}{2}\left\{\sup\left[a;\ T_n(a,\hat{\nu}_{1n}^{(R)})>0\right]+\inf\left[a;\ T_n(a,\hat{\nu}_{1n}^{(R)})<0\right]\right\}. \tag{5.6}$$

Now, recall that F^* is symmetric about 0 so that $T_n(0,\nu_1)$ is symmetric about 0 and we obtain (see Saleh and Sen (1978))

$$\sup\left\{n^{1/2}\left|T_n\left(n^{-1/2}t_1,\nu_1+n^{-1/2}t_2\right)-T_n(0,\nu_1)+\gamma(\varphi,f^*)\left[t_1+t_2\mu_x\right]\right|\right\}=o_p(1), \tag{5.7}$$

where the supremum is taken over $|t_1|\le c_1$ and $t_2\le c_2$. This together with (4.15) implies that we can use the statistics

$$L_n^*(\nu_1)=n^{-1/2}\sum_{i=1}^{n}(x_i^0-\bar{x}_n^0)\varphi(F^*(z_i))+o_p(1) \tag{5.8}$$

and

$$T_n^*(0,\nu_1)=n^{-1/2}\sum_{i=1}^{n}\varphi^{+}(F^*(z_i))\mathsf{sgn}(z_i)+o_p(1) \tag{5.9}$$

to represent the estimator of ν_0 as

$$\sqrt{n}(\hat{\nu}_{0n}^{(R)}-\nu_0)=\gamma^{-1}(\varphi,f^*)\left[T_n^*(0,\nu_1)-\mu_xL_n^0(\nu_1)\right]+o_p(1), \tag{5.10}$$

since

$$\begin{pmatrix}T_n(0,\nu_1)\\ L_n(\nu_1)\end{pmatrix}\sim\mathsf{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix},\ A_\varphi^2\begin{pmatrix}1&0\\0&\lambda_x^2\sigma_x^2+\lambda_u^2\sigma_u^2\end{pmatrix}\right), \tag{5.11}$$

and we have

$$\begin{pmatrix}\sqrt{n}(\hat{\beta}_{0n}^{(R)}-\beta_0)\\ \sqrt{n}(\hat{\beta}_{1n}^{(R)}-\beta_1)\end{pmatrix}\sim\mathsf{N}_2\left(\begin{pmatrix}0\\0\end{pmatrix},\ \frac{A_\varphi^2}{\gamma^2(\varphi,f^*)}\begin{pmatrix}1+\frac{\mu_x^2}{\kappa_x\lambda_x^2\sigma_x^2}&-\frac{\mu_x^2}{\kappa_x\lambda_x^2\sigma_x^2}\\ -\frac{\mu_x^2}{\kappa_x\lambda_x^2\sigma_x^2}&\frac{1}{\kappa_x\lambda_x^2\sigma_x^2}\end{pmatrix}\right). \tag{5.12}$$

Hence, the JARE of $\hat{\boldsymbol{\beta}}_n^{(R)}$ compared to $\boldsymbol{\beta}_n^{*(R)}$ is given by

$$\mathsf{JARE}(\hat{\boldsymbol{\beta}}_n^{(R)}:\boldsymbol{\beta}_n^{*(R)})=\kappa_x\frac{\gamma^2(\varphi,f^*)}{\gamma^2(\varphi,f)}=\kappa_x\frac{B^2(F^*)}{B^2(F)} \tag{5.13}$$

and the JARE of $\hat{\boldsymbol{\beta}}_n^{(R)}$ compared to the LSE is given by

$$48\sigma_e^2(1+\Delta^{*2})G^*B^2(F^*) \tag{5.14}$$

if Wilcoxon's score is used in both cases.

6 Jaeckel's Estimator of Slope

In this section we consider the Jaeckel's (1972) estimator of the slope ν_1 of the model (3.11). The dispersion $D(z)$ of $z_1, \ldots, z_n$ is minimized, where

$$z_i = y_i - \nu_1 x_i^0, \ i = 1, \ldots, n. \tag{6.1}$$

Accordingly, let $a_n^{\varphi}(i), \ i = 1, \ldots, n$, be a non-decreasing set of scores, not all equal, satisfying the condition

$$\sum_{i=1}^{n} a_n^{\varphi}(i) = 0. \tag{6.2}$$

Based on the observations $\{(x_i^0, y_i)' | i = 1, \ldots, n\}$ from (3.16), consider $z_{(1)} \leq \cdots \leq z_{(n)}$ to be the ordered values of the residuals (6.1). Let us write $z_{(k)} = y_{i(k)} - \nu_1 x_{i(k)}^0$, where $i(k)$ is the index of the observations giving rise to the k-th ordered residuals. The dispersion of residuals as a function of ν_1 is then given by

$$D(y - \nu_1 x^0) = \sum_{k=1}^{n} a_n^{\varphi}(k) \left[y_{i(k)} - \nu_1 x_{i(k)}^0 \right]. \tag{6.3}$$

By Theorem 1 of Jaeckel (1972), $D(y - \nu_1 x^0)$ is a non-negative, continuous and convex function of ν_1. The minimization of (6.3) with respect to β_1 yields the R-estimator of ν_1. But the estimator may not be unique. Thus, any choice of ν_0 is acceptable. There is an asymptotic equivalence between the Jaeckel's estimator using (6.3) and the estimators discussed in Section 3 and 4; these solve the same equation since

$$\frac{\partial}{\partial \Delta} D(y - \Delta x^0) = -L_n(y - \Delta x^0), \quad \text{where } \Delta = n^{1/2}\nu_1. \tag{6.4}$$

In other words the derivative of the Jaeckel's dispersion criterion is the negative of the linear rank statistics subject to (6.2). Let $\hat{\nu}_{1n}^{(J)}$ denote the consistent Jaeckel's estimator of ν_1, obtained by minimizing (6.3). Then, the R-estimator is obtained as:

$$\hat{\beta}_{1n}^{(J)} = \hat{\kappa}_x^{-1} \hat{\nu}_{1n}^{(J)} = [1 - (S_{nxx} + S_{nuu})^{-1} S_{nuu}]^{-1} \hat{\nu}_{1n}^{(J)}. \tag{6.5}$$

Now considering the AUL result given at (4.7) and (4.16), we investigate the quadratic form

$$Q(\delta) = \gamma(\varphi, f^*)(S_{nxx} + S_{nuu})\delta^2 - L_n^*(\beta_1)\delta + O_p(1). \tag{6.6}$$

The derivative of $Q(\delta)$ with respect to δ is given by

$$\frac{\partial Q(\delta)}{\partial \delta} = \gamma(\varphi, f^*)(S_{nxx} + S_{nuu})\delta - L_n^*(\nu_1), \tag{6.7}$$

where $L_n^*(\nu_1)$ is given by (4.17). This leads to the asymptotic representation of

$$\sqrt{n}(\hat{\nu}_{1n}^{(J)} - \nu_1) = \gamma^{-1}(\varphi, f^*)(S_{nxx} + S_{nuu})^{-1} L_n^*(\nu_1) + o_p(1), \tag{6.8}$$

yielding the asymptotic distribution of $\sqrt{n}(\hat{\nu}_{1n}^{(J)} - \nu_1)$ as

$$\mathsf{N}\left(0,\ \left[\frac{A_\varphi^2}{\gamma^2(\varphi, f^*)}\right][\sigma_{x^0}^2]^{-1}\right) \tag{6.9}$$

as given in (4.18). Hence,

$$\sqrt{n}(\hat{\beta}_{1n}^{(J)} - \beta_1) \sim \mathsf{N}\left(0,\ \left[\frac{A_\varphi^2}{\gamma^2(\varphi, f^*)}\right](\kappa_x \lambda_x^2 \sigma_x^2)^{-1}\right). \tag{6.10}$$

We now consider the Jaeckel's (1972) estimator using Wilcoxon scores given by

$$a_n^W(i) = \frac{i}{n+1} - \frac{1}{2}, \quad i = 1, \ldots, n. \tag{6.11}$$

Then, the Jaeckel's estimator based on the residuals (6.1) is given by the median of the divided differences $\left\{(y_j - y_i)/(x_j^0 - x_i^0) | 1 \le i < j \le n\right\}$, i.e.

$$\hat{\nu}_{1n}^{(J)} = \underset{1 \le i < j \le n}{\text{med}} \frac{y_j - y_i}{x_j^0 - x_i^0}. \tag{6.12}$$

Consequently, the slope estimator is given by

$$\hat{\beta}_{1n}^{(J)} = \hat{\kappa}_x^{-1}\left\{\underset{1 \le i < j \le n}{\text{med}} \frac{y_j - y_i}{x_j^0 - x_i^0}\right\}, \tag{6.13}$$

where $\hat{\kappa}_x$ is given by (3.17). Thus, it is easily proved using (6.8) that

$$\sqrt{n}(\hat{\nu}_{1n}^{(J)} - \nu_1) \sim \mathsf{N}\left(0,\ \frac{1}{48(\int f^{*2}(x)dx)^2 \sigma_{x^0}^2}\right) \tag{6.14}$$

and

$$\sqrt{n}(\hat{\beta}_{1n}^{(J)} - \beta_1) \sim \mathsf{N}\left(0,\ \frac{1}{48(\int f^{*2}(x)dx)^2 \kappa_x \lambda_x^2 \sigma_x^2}\right), \tag{6.15}$$

The ARE of the estimator compared to the least squares estimator is clearly given by

$$48(\sigma_e^2 + \kappa_x \lambda_u^2 \sigma_u^2 \beta_1^2)\left(1 + \frac{\kappa_x \beta_1^2 E[u^4 - 2\sigma_u^4]}{(\sigma_e^2 + \kappa_x \lambda_u^2 \sigma_u^2 \beta_1^2)\lambda_x^2 \sigma_x^2}\right) B^2(F^*). \tag{6.16}$$

And the ARE of $\hat{\beta}_{1n}^{(J)}$ compared to $\beta^{*(J)}$ is given by

$$\kappa_x \frac{\gamma^2(\varphi, f^*)}{\gamma^2(\varphi, f)} = \kappa_x \frac{B^2(F^*)}{B^2(F)} \le \kappa_x B^2(H_\theta). \tag{6.17}$$

We may remind the readers that Jaeckel's estimator $\hat{\nu}_{1n}^{(J)}$ and $\hat{\beta}_{1n}^{(J)}$ may also be obtained using Kendall's tau test, resulting in the Theil-Sen type estimator (Sen 1968) in the model (3.1) with the ME version in Sen and Saleh (2010). In the next section we provide some simulation studies on the performance of the R-estimators for the ME models.

7 Simulation Study

We have conducted a simulation study to check how the proposed estimation procedures perform for finite sample situations. We considered the model

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + e_i, && i = 1, \ldots, n, \\ x_i^0 &= x_i + u_i, && i = 1, \ldots, n, \end{aligned} \tag{7.1}$$

where the errors $e_i,\ i = 1, \ldots, n$, were simulated from the normal $\mathsf{N}(0,1)$, Laplace $\mathsf{L}(0,1)$ and Cauchy distributions. The measurement errors $u_i,\ i = 1, \ldots, n$ were generated independently from the normal $\mathsf{N}(0,0.5)$, $\mathsf{N}(0,2)$ and uniform $\mathsf{U}(-1,1)$ distributions. We considered vectors with design points $x_1, \ldots, x_n$ generated from the uniform distribution on the interval (-2,10).

The following parameter values of the model were used:

- sample sizes: $n = 20,\ 100$;
- $\beta_0 = 1,\ \beta_1 = 3$;
- Wilcoxon score function $\varphi(u) = 2u - 1,\ 0 \le u \le 1$, and $A_\varphi^2 = \frac{1}{12}$.

10 000 replications of the model were simulated for each combination of the parameters and a particular distribution of the measurement errors. The slope parameter was estimated by the least squares, R- and Jaeckel's estimators. For the sake of comparison, the mean square error (MSE), mean, median, 2.5%– and 97.5%–sample quantiles were computed and summarized in Tables 1-9. These tables compare the slope estimators under different conditions of the sample sizes, error distributions and measurement errors. These results are reproduced from the paper by Saleh et al (2009).

Conclusion

The simulation study indicates:

(i) The R-estimator and Jaeckel's estimator give very good results, though intuitively the least squares estimator would be more favorable for the normal distribution.

(ii) The influence of even a small sample size is not too big.

(iii) The R-estimator and Jaeckel's estimator show a good performance for the measurement errors with a small variance.

We have made more extensive simulation experiments. Particularly, we considered various score functions, design vectors, other underlying distributions of the error terms and the measurement errors with a small variance. The corresponding estimates have appeared to be very slightly influenced by these parameters.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00466	2.99942	2.8647	2.99924	3.13477
	N(0, 0.5)	0.02585	3.00118	2.68445	2.99926	3.31555
	N(0, 2)	0.08827	3.00094	2.41449	2.99593	3.59150
	U(−1, 1)	0.01838	2.99971	2.73392	3.00007	3.26213
R	–	0.00507	2.99913	2.86009	2.99834	3.14291
	N(0, 0.5)	0.02777	3.00045	2.67288	2.99786	3.32993
	N(0, 2)	0.09290	3.00336	2.41420	2.99910	3.60857
	U(−1, 1)	0.01798	2.99558	2.73435	2.99541	3.25841
Jaeckel	–	0.00529	2.99920	2.85672	2.99907	3.14351
	N(0, 0.5)	0.02921	3.01801	2.67955	3.01528	3.36205
	N(0, 2)	0.09979	3.04651	2.44197	3.03935	3.65707
	U(−1, 1)	0.01868	3.00963	2.74470	3.01258	3.27215

Table 1: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 20$ and the standard normal distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00899	3.00142	2.81441	3.00100	3.19353
	N(0, 0.5)	0.02970	3.00438	2.67894	3.00094	3.35343
	N(0, 2)	0.09093	3.00475	2.41145	3.00343	3.58918
	U(−1, 1)	0.02284	3.00298	2.70889	3.00071	3.30307
R	–	0.00707	3.00088	2.83494	3.00134	3.17287
	N(0, 0.5)	0.03101	3.00327	2.66215	2.99926	3.34867
	N(0, 2)	0.09622	3.00468	2.39988	3.00659	3.61228
	U(−1, 1)	0.02238	2.99949	2.71051	2.99607	3.29889
Jaeckel	–	0.00733	3.00069	2.82944	3.00115	3.17347
	N(0, 0.5)	0.03287	3.02051	2.67842	3.01766	3.38284
	N(0, 2)	0.10324	3.04708	2.44339	3.04161	3.67898
	U(−1, 1)	0.02332	3.01260	2.71744	3.00966	3.31475

Table 2: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 20$ and the Laplace distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	868.7410	2.99596	-0.19920	2.99559	6.83956
	N(0, 0.5)	847.6568	2.95700	-0.31639	2.99069	6.82395
	N(0, 2)	711.9615	3.08034	-0.39872	2.98669	6.92081
	U(−1, 1)	969.4066	3.13333	-0.20596	2.99226	6.91857
R	–	0.02435	2.99856	2.67390	3.00072	3.30730
	N(0, 0.5)	0.06643	2.99559	2.50962	2.99584	3.52798
	N(0, 2)	0.16673	2.99866	2.22365	2.99282	3.82022
	U(−1, 1)	0.05206	2.99296	2.54776	2.99075	3.44513
Jaeckel	–	0.02301	2.99877	2.68850	3.00054	3.29576
	N(0, 0.5)	0.06817	3.01331	2.51926	3.01089	3.54477
	N(0, 2)	0.17318	3.03930	2.26856	3.03128	3.89083
	U(−1, 1)	0.05244	3.00637	2.56401	3.00336	3.46739

Table 3: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \dots, n$, the sample size $n = 20$ and the Cauchy distribution of errors $e_i,\ i = 1, \dots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00072	3.00033	2.94677	3.00063	3.05131
	N(0, 0.5)	0.00413	2.99901	2.87156	2.99939	3.12368
	N(0, 2)	0.01388	2.99804	2.76341	2.99731	3.23099
	U(−1, 1)	0.00308	3.00071	2.89090	3.00068	3.10848
R	–	0.00077	3.00033	2.94422	3.00056	3.05302
	N(0, 0.5)	0.00431	2.99983	2.87180	3.00055	3.12718
	N(0, 2)	0.01422	3.00303	2.76645	3.00133	3.23695
	U(−1, 1)	0.00289	2.99536	2.89034	2.99591	3.09981
Jaeckel	–	0.00077	3.00035	2.94462	3.00051	3.05388
	N(0, 0.5)	0.00435	3.00019	2.87142	3.00058	3.12858
	N(0, 2)	0.01464	3.01656	2.78057	3.01529	3.25044
	U(−1, 1)	0.00290	2.99468	2.88935	2.99486	3.09859

Table 4: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \dots, n$, the sample size $n = 100$ and the standard normal distribution of errors $e_i,\ i = 1, \dots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00150	2.99934	2.92448	2.99926	3.07669
	N(0, 0.5)	0.00488	2.99973	2.86708	2.99799	3.13954
	N(0, 2)	0.01528	2.99737	2.75938	2.99331	3.24431
	U(−1, 1)	0.00379	2.99837	2.87889	2.99952	3.11731
R	–	0.00104	2.99934	2.93654	2.99902	3.06413
	N(0, 0.5)	0.00495	2.99992	2.86580	2.99885	3.14055
	N(0, 2)	0.01569	3.00142	2.76264	2.99948	3.25023
	U(−1, 1)	0.00355	2.99381	2.87555	2.99379	3.10981
Jaeckel	–	0.00104	2.99932	2.93578	2.99904	3.06419
	N(0, 0.5)	0.00501	3.00058	2.86546	2.99871	3.14310
	N(0, 2)	0.01608	3.01611	2.77713	3.01421	3.26630
	U(−1, 1)	0.00359	2.99324	2.87548	2.99332	3.10976

Table 5: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 100$ and the Laplace distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	420.1448	2.89912	0.40932	3.01662	6.65968
	N(0, 0.5)	479.1673	2.90735	0.33433	3.01176	6.82481
	N(0, 2)	557.7343	2.83851	0.09792	3.01329	6.72500
	U(−1, 1)	444.3006	2.90010	0.19087	3.01646	6.73704
R	–	0.00267	3.00122	2.89646	3.00212	3.10032
	N(0, 0.5)	0.00934	3.00194	2.80682	3.00201	3.19296
	N(0, 2)	0.02352	3.00801	2.70963	3.00701	3.30800
	U(−1, 1)	0.00720	3.00092	2.83388	3.00114	3.16662
Jaeckel	–	0.00266	3.00129	2.89643	3.00195	3.10059
	N(0, 0.5)	0.00947	3.00284	2.80988	3.00400	3.19459
	N(0, 2)	0.02428	3.02411	2.72223	3.02127	3.32601
	U(−1, 1)	0.00724	3.00017	2.83262	3.00034	3.16889

Table 6: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 100$ and the Cauchy distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00016	3.00002	2.97457	3.00018	3.02573
	N(0, 0.5)	0.00090	2.99996	2.94132	3.0002	3.05877
	N(0, 2)	0.00303	2.99939	2.89386	2.99921	3.10636
	U(−1, 1)	0.00067	2.99980	2.94866	2.99968	3.05169
R	–	0.00017	3.00003	2.97414	3.00015	3.02639
	N(0, 0.5)	0.00095	3.00037	2.94061	3.00046	3.05976
	N(0, 2)	0.00317	3.00477	2.89686	3.00520	3.11546
	U(−1, 1)	0.00065	2.99397	2.94454	2.99371	3.04362
Jaeckel	–	0.00017	3.00003	2.97416	3.00010	3.02625
	N(0, 0.5)	0.00097	3.00455	2.94461	3.00499	3.06461
	N(0, 2)	0.00391	3.02762	2.92018	3.02778	3.13775
	U(−1, 1)	0.00063	2.99626	2.94757	2.99600	3.04520

Table 7: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 500$ and the standard normal distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	0.00031	2.99972	2.96607	2.99971	3.03374
	N(0, 0.5)	0.00106	2.99910	2.93580	2.99858	3.06320
	N(0, 2)	0.00323	3.00045	2.88965	3.00101	3.11198
	U(−1, 1)	0.00082	3.00004	2.94431	2.9998	3.05559
R	–	0.00021	2.99972	2.97154	2.99989	3.02772
	N(0, 0.5)	0.00108	2.99933	2.93513	2.99869	3.06431
	N(0, 2)	0.00335	3.00570	2.89365	3.00588	3.11936
	U(−1, 1)	0.00077	2.99501	2.94133	2.99478	3.04784
Jaeckel	–	0.00021	2.99973	2.97157	2.99983	3.02763
	N(0, 0.5)	0.00109	3.00366	2.93983	3.00329	3.06801
	N(0, 2)	0.00419	3.02911	2.91569	3.02958	3.14090
	U(−1, 1)	0.00075	2.99730	2.94341	2.99733	3.05056

Table 8: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \ldots, n$, the sample size $n = 500$ and the Laplace distribution of errors $e_i,\ i = 1, \ldots, n$.

Estimator	u_j	MSE	mean	2.5%-q.	median	97.5%-q.
LSE	–	8576.803	1.58670	-0.03652	2.99776	6.11806
	N(0, 0.5)	11153.72	1.27588	-0.16233	2.99874	6.12812
	N(0, 2)	11885.26	1.45589	-0.30151	2.99726	6.47853
	U(−1, 1)	5778.837	1.98624	-0.11801	2.99830	6.34738
R	–	0.00055	2.99975	2.95469	2.99948	3.04758
	N(0, 0.5)	0.00193	3.00009	2.91503	2.99994	3.08592
	N(0, 2)	0.00501	3.00425	2.86343	3.00355	3.13989
	U(−1, 1)	0.00151	2.99562	2.92148	2.99570	3.07219
Jaeckel	–	0.00055	2.99977	2.95457	2.99938	3.04754
	N(0, 0.5)	0.00195	3.00465	2.91823	3.00444	3.09070
	N(0, 2)	0.00589	3.02899	2.88856	3.02881	3.16543
	U(−1, 1)	0.00150	2.99794	2.92347	2.99797	3.07405

Table 9: Sample statistics of 10 000 values of the estimated slope parameter in model (7.1) for the Least Squares, the R- and the Jaeckel's estimators, various distributions of the measurement errors $u_j,\ j = 1, \dots, n$, the sample size $n = 500$ and the Cauchy distribution of errors $e_i,\ i = 1, \dots, n$.

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