Journal of Statistical Research *ISSN 0256 - 422 X*
2010, Vol. 44, No. 2, pp. 315-334
Bangladesh

ON APPROXIMATING THE DISTRIBUTIONS OF RATIOS AND DIFFERENCES OF NONCENTRAL QUADRATIC FORMS IN NORMAL VECTORS

Aliakbar Mohsenipour
Department of Statistical & Actuarial Sciences
The University of Western Ontario, London, Ontario, Canada, N6A 5B7
Email: amohsen@uwo.ca

Serge B. Provost
Department of Statistical & Actuarial Sciences
The University of Western Ontario, London, Ontario, Canada, N6A 5B7
Email: provost@stats.uwo.ca

SUMMARY

The distribution of positive definite quadratic forms in normal random vectors is first approximated by generalized gamma and Pearson-type density functions. The distribution of indefinite quadratic forms is then obtained from their representation in terms of the difference of two positive definite quadratic forms. In the case of the Pearson-type approximant, explicit representations are obtained for the density and distribution functions of an indefinite quadratic form. A moment-based technique whereby the initial approximations are adjusted by means of polynomials is being introduced. A detailed algorithm describing the steps involved in the methodology advocated herein is provided as well. It is also explained that the distributional results apply to the ratios of certain quadratic forms. Two numerical examples are presented: the first involves an indefinite quadratic form while the second approximates the distribution of the Durbin-Watson statistic, which is shown to be expressible as a ratio of quadratic forms.

Keywords and phrases: Indefinite Quadratic Forms, Density Approximation, Ratios, Simulations, Durbin-Watson Statistic.

AMS Classification: Primary: 62E17; Secondary: 62M10.

1 Introduction

Numerous distributional results are already available in connection with quadratic forms in normal random variables and ratios thereof. Various representations of the density function of a quadratic form have been derived and several procedures have been proposed for computing percentage points. Box (1954) considered a linear combination of chi-square variables having even degrees of freedom. Gurland (1953), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961), and Kotz *et al.* (1967a,b) among others, obtained expressions involving MacLaurin series and the distribution function of chi-square variables. Gurland (1956) and Shah (1963) considered respectively central and noncentral indefinite quadratic forms, but as pointed by Shah (1963), the expansions obtained are not practical. Exact distributional results were derived by Imhof (1961), Davis (1973) and Rice (1980).

As pointed out in Mathai and Provost (1992), a wide array of test statistics can be expressed in terms of quadratic forms in normal random vectors. For example, one may consider the lagged regression residuals developed by De Gooijer and MacNeill (1999) and discussed in Provost *et al.* (2005), or certain change point test statistics derived by MacNeill (1978).

An accessible approach is proposed in this paper for approximating the densities of positive definite and indefinite quadratic forms in normal random variables from gamma, generalized gamma and Pearson-type densities. It is explained that such approximants can be combined with polynomial adjustments in order to improve their accuracy. These results can also be utilized to determine the approximate distributions of the ratios of certain quadratic forms. Such ratios arise for example in regression theory, linear models, analysis of variance and time series. For instance, the sample serial correlation coefficient as defined in Anderson (1990) and discussed in Provost and Rudiuk (1995) as well as the sample innovation cross-correlation function for an ARMA time series whose asymptotic distribution was derived by McLeod (1979), have such a structure. Koerts and Abrahamse (1969) investigated the distribution of ratios of quadratic forms in the context of the general linear model. Shenton and Johnson (1965) derived the first few terms of the series expansions of the first and second moments of the sample circular serial correlation coefficient. Inder (1986) developed an approximation to the null distribution of the Durbin-Watson statistic to test for autoregressive disturbances in a linear regression model with a lagged dependent variable and determined its critical values. This test statistic can in fact be expressed as a ratio of quadratic forms wherein the matrix of the quadratic form appearing in the denominator is idempotent.

The Monte Carlo and analytical approaches have their own merits and shortcomings. Monte Carlo simulations which generate artificial data wherefrom sampling distributions and moments are estimated, can be implemented and brought to bear with relative ease on an extensive range of models and error probability distributions. There are, however, some limitations on the range of applicability of these experiments: the results may be subject to sampling variations or simulation inadequacies and may depend on the assumed parameter values. Recent efforts to cope with these issues are discussed for example in Hendry (1979), Hendry and Harrison (1974), Hendry and Mizon (1980) and Dempster *et al.* (1977). The analytical approach, on the other hand, derives results which hold over the whole parameter space but may find limitations in terms of simplifications on the model, which have to be imposed to make the problem tractable. Even when exact theoretical results can be obtained, the resulting expressions can be fairly complicated. The moment-based approximation procedure advocated in this paper has the advantage of producing closed form expressions that yield very accurate results over the entire supports of the distributions being considered.

A representation of noncentral indefinite quadratic forms which results from an application of the spectral decomposition theorem is given in terms of the difference of two positive definite quadratic forms in Section 2; a formula for determining their moments from their cumulants as well as an integral representation of the density function of an indefinite quadratic form are also provided therein. Pearson and gamma-types approximations to the distribution of positive definite quadratic forms are introduced in Section 3; explicit representations of the approximate density and distribution functions of an indefinite quadratic form are given in the case of the Pearson-type approximation and a moment-based technique for improving the approximations by means of polynomial adjustments is then presented. An algorithm describing the methodology is provided in Section 4. The proposed density approximation technique is applied to an indefinite quadratic form and the Durbin-Watson statistic in Section 5.

2 Noncentral Indefinite Quadratic Forms

In this section, a decomposition of noncentral indefinite quadratic forms is given in terms of the difference of two positive definite quadratic forms whose moments are determined from a certain recursive relationship involving their cumulants. An integral representation of the density function of an indefinite quadratic form is also provided.

Indefinite quadratic form in normal random variables can be expressed in terms of standard normal variables as follows. Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, that is, $\mathbf{X}$ is distributed as a p-variate normal random vector with mean $\boldsymbol{\mu}$ and positive definite covariance matrix Σ. On letting $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$, where I is a $p \times p$ identity matrix, one has $\mathbf{X} = \Sigma^{\frac{1}{2}}\mathbf{Z} + \boldsymbol{\mu}$ where $\Sigma^{\frac{1}{2}}$ denotes the symmetric square root of Σ. Then, in light of the spectral decomposition theorem, the quadratic form $Q = \mathbf{X}'A\mathbf{X}$ where A is a $p \times p$ real symmetric matrix and $\mathbf{X}'$ denotes the transpose of $\mathbf{X}$, can be expressed as follows:

$$
\begin{aligned}
Q &= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}) \\
&= (\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})'PP'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}PP'(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu})\,, \qquad (2.1)
\end{aligned}
$$

where P is an orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$, that is, $P'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}P = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1, \ldots, \lambda_p$ being the eigenvalues of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ (or equivalently those of $A\Sigma$) in *decreasing* order. Let $\mathbf{v}_i$ denote the *normalized* eigenvector of $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}$ *corresponding* to λ_i, $i = 1, \ldots, p$, (such that $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ *and* $\mathbf{v}_i{}'\mathbf{v}_i = 1$) and $P = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$. Letting $\mathbf{U} = P'\mathbf{Z}$, $\mathbf{U} \sim \mathcal{N}_p(\mathbf{0}, I)$ since P is an orthogonal matrix, and then, one has

$$
\begin{aligned}
Q &= (\mathbf{U} + \mathbf{b})'\mathrm{Diag}(\lambda_1, \ldots, \lambda_p)(\mathbf{U} + \mathbf{b}) \\
&= \sum_{j=1}^{p} \lambda_j (U_j + b_j)^2\,, \qquad (2.2)
\end{aligned}
$$

where $\text{Diag}(\lambda_1, \ldots, \lambda_p)$ is a diagonal matrix whose diagonal elements are $\lambda_1, \ldots, \lambda_p$, $\mathbf{b} = P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu}$ with $\mathbf{b} = (b_1, \ldots, b_p)'$, $\mathbf{U} = (U_1, \ldots, U_p)'$, and $(U_j + b_j)$, $j = 1, \ldots, p$, are independently distributed $\mathcal{N}\ (b_j, 1)$ random variables. It follows that

$$\begin{aligned} Q &= \sum_{j=1}^{r} \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| (U_j + b_j)^2 \\ &\equiv Q_1 - Q_2\,, \end{aligned} \tag{2.3}$$

where r is the number of positive eigenvalues of $A\Sigma$ and $p - r - \theta$ is the number of negative eigenvalues of $A\Sigma$, θ being the number of null eigenvalues. Thus, a noncentral indefinite quadratic form, Q, can be expressed as a difference of independently distributed linear combinations of independent non-central chi-square random variables having one degree of freedom each, or equivalently, as the difference of two positive definite quadratic forms. It should be noted that the chi-square random variables are central whenever $\boldsymbol{\mu} = \mathbf{0}$. When the matrix A is positive semidefinite, so is Q, and then, $Q \sim Q_1$ as defined in Equation (2.3). Moreover, if A is not symmetric, it suffices to replace this matrix by $(A + A')/2$ in a quadratic form. Accordingly, it will be assumed without any loss of generality that the matrices of the quadratic forms being considered are symmetric.

The moments of a quadratic forms, which are useful for estimating the parameters of the density approximants, can be determined as follows. As shown in Mathai and Provost (1992), the s^{th} cumulant of $\mathbf{X}'A\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ is

$$\begin{aligned} k(s) &= 2^{s-1} s! \Big(\text{tr}(A\,\Sigma)^s / s \ + \ \boldsymbol{\mu}'(A\Sigma)^{s-1} A\boldsymbol{\mu} \Big) \\ &= 2^{s-1}(s-1)!\ \theta_s\,, \end{aligned} \tag{2.4}$$

where $\text{tr}(\cdot)$ denotes the trace of $(\cdot)$ and $\theta_s = \sum_{j=1}^{p} \lambda_j^s (1 + s\, b_j^2),\ \ s = 1, 2, \ldots$. It should be noted that $\text{tr}(A\Sigma)^s = \sum_{j=1}^{p} \lambda_j^s$ where the λ_j's, $j = 1, \ldots, p$, are the eigenvalues of $A\Sigma$. The h^{th} moment of $\mathbf{X}'A\mathbf{X}$ can be obtained from its cumulants by means of the following recursive relationship, which was derived by for instance by Smith (1995):

$$\mu(h) \ = \ \sum_{i=0}^{h-1} \frac{(h-1)!}{(h-1-i)!\, i!}\, k(h-i)\, \mu(i)\,, \tag{2.5}$$

where $k(s)$ is as given in Equation (2.4).

One can make use of Equation (2.5) to determine the moments of each of the positive definite quadratic forms, $Q_1 \equiv \mathbf{W}_1' A_1 \mathbf{W}_1$ and $Q_2 \equiv \mathbf{W}_2' A_2 \mathbf{W}_2$, appearing in Equation (2.3) where $A_1 = \text{diag}(\lambda_1, \ldots, \lambda_r)$, $A_2 = \text{diag}(|\lambda_{r+\theta+1}|, \ldots, |\lambda_p|)$, $\mathbf{W}_1 \sim \mathcal{N}_r(\boldsymbol{b_1}, I)$, $\boldsymbol{b_1} = (b_1, \ldots, b_r)'$, and $\mathbf{W}_2 \sim \mathcal{N}_{p-r-\theta}(\boldsymbol{b_2}, I)$, $\boldsymbol{b_2} = (b_{r+\theta+1}, \ldots, b_p)'$, the b_j's being as defined in Equation (2.2).

Since an indefinite quadratic form is distributed as the difference of two positive definite quadratic forms, its density function can be obtained via the transformation of variables technique. For the problem at hand, letting $h_Q(q)\,\mathcal{I}_{\Re}(x)$, $f_{Q_1}(q_1)\,\mathcal{I}_{(\tau_1,\infty)}(x)$ and

$f_{Q_2}(q_2)\,\mathcal{I}_{(\tau_2,\infty)}(x)$ respectively denote the approximate densities of Q, Q_1 and Q_2, where the $\mathcal{I}_{\mathcal{S}}(.)$ is the indicator function with respect to the set $\mathcal{S}$, an approximation to the density function of the indefinite quadratic form Q can then be obtained as follows:

$$h_Q(q) = \begin{cases} h_P(q) & \text{for } q \geq \tau_1 - \tau_2 \\ h_N(q) & \text{for } q < \tau_1 - \tau_2, \end{cases} \tag{2.6}$$

where

$$h_P(q) = \int_{q+\tau_2}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\,\mathrm{d}y \tag{2.7}$$

and

$$h_N(q) = \int_{\tau_1}^{\infty} f_{Q_1}(y) f_{Q_2}(y-q)\,\mathrm{d}y\,. \tag{2.8}$$

Noting that

$$\Pr\Big(\frac{\mathbf{X}'A\mathbf{X}}{\mathbf{X}'B\mathbf{X}} < t_0\Big) = \Pr(\mathbf{X}'(A - t_0\,B)\mathbf{X} < 0)\,, \tag{2.9}$$

it is seen that the distribution of the ratio of quadratic forms, $\mathbf{X}'A\mathbf{X}/\mathbf{X}'B\mathbf{X}$, can readily be determined from that of an indefinite quadratic form.

3 Various Approximations

3.1 Approximation via Pearson's Approach

Let $E(Q_i)$ and σ_{Q_i} denote the mean and standard deviation of the positive definite quadratic form Q_i. According to Pearson (1959), one can write $Q_i \approx U_i$ with

$$U_i \sim \Big(\frac{\chi^2_{\nu_i} - \nu_i}{\sqrt{2\nu_i}}\Big)\,\sigma_{Q_i} + E(Q_i)\,, \tag{3.1}$$

where the symbol $\approx$ means "is approximately distributed as" and ν_i is such that both Q_i and U_i have equal third cumulants. Since $E(\chi^2_{\nu_i}) = \nu_i$ and $Var(\chi^2_{\nu_i}) = 2\nu_i$, $E(U_i) = E(Q_i)$ and $Var(U_i) = \sigma^2_{Q_i}$. Letting θ_i be as defined in (2.4), the third cumulant of U_i is

$$8\nu_i\,\sigma^3_{Q_i}/(2\nu_i)^{3/2} = 2^{3/2}k_i(2)^{3/2}/\sqrt{\nu_i} = 8\theta_2^{3/2}/\sqrt{\nu_i},$$

while the first and second cumulants of U_i coincide with those of Q_i. On equating the third cumulants of U_i and Q_i, which according to (2.3) is $8\ \theta_3$, one has

$$\nu_i = \theta_2^3/\theta_3^2\,. \tag{3.2}$$

Thus,

$$Q_i \approx \frac{\theta_3}{\theta_2}\,\chi^2_{\nu_i} - \frac{\theta_2^3}{\theta_3^2} + \theta_1 \tag{3.3}$$

or equivalently,

$$Q_i \approx c_i\,\chi^2_{\nu_i} + \tau_i, \tag{3.4}$$

where $c_i = \theta_3/\theta_2$ and $\tau_i = \theta_1 - \theta_2^3/\theta_3^2$. That is, Pearson's approximant to the exact density of Q_i is given by

$$f_{Q_i}(q_i) \quad = \quad \frac{(q_i - \tau_i)^{\nu_i/2-1}\, e^{-(q_i-\tau_i)/(2c_i)}}{\Gamma(\frac{\nu_i}{2})(2c_i)^{\nu_i/2}}\, \mathcal{I}_{(\tau_i,\infty)}(q_i)\,. \tag{3.5}$$

Accordingly, the density function of the indefinite quadratic form $Q = Q_1 - Q_2$, where Q_1 and Q_2 are positive definite quadratic forms, can be approximated by making use of Equation (2.6) where $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$ respectively denote the Pearson-type density approximants of Q_1 and Q_2 which are available from Equation (3.5). Explicit representations of $h_P(q)$ and $h_N(q)$ as specified by Equations (2.7) and (2.8), respectively, can be derived as follows:

$$\begin{aligned}
h_N(q) &= \int_{\tau_1}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y \\
&= \int_{\tau_1}^{\infty} \frac{(y-\tau_1)^{\nu_1/2-1}\,(y-q-\tau_2)^{\nu_2/2-1}\, e^{-(y-\tau_1)/(2c_1)}\, e^{-(y-q-\tau_2)/(2c_2)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\nu_1/2}\,(2c_2)^{\nu_2/2}}\,\mathrm{d}y \\
&= \frac{(c_1)^{-\nu_1/2}\,(c_2)^{-\nu_2/2}\, e^{(-\tau_1+\tau_2+q)/(2c_2)}}{(\tau_1-\tau_2-q)\,\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_2}{2})\,\Gamma(\frac{\nu_2}{2})}\left(\frac{c_1\,c_2}{c_1+c_2}\right)^{(\nu_1+\nu_2)/2}\Bigg(2^{(-\nu_1-\nu_2)/2}\,\Gamma\Big(\frac{\nu_1}{2}\Big) \\
&\quad\times\ \Gamma\Big(-\frac{\nu_1}{2}-\frac{\nu_2}{2}+1\Big)\,{}_1F_1\Big(\frac{\nu_1}{2};\frac{\nu_1+\nu_2}{2};\frac{(c_1+c_2)\,(\tau_1-\tau_2-q)}{2c_1\,c_2}\Big) \\
&\quad\times\ \Big(\frac{(c_1+c_2)\,(\tau_1-\tau_2-q)}{c_1\,c_2}\Big)^{(\nu_1+\nu_2)/2} + \frac{1}{2c_1\,c_2}\,(c_1+c_2)\,(\tau_1-\tau_2-q) \\
&\quad\times\ \Gamma\Big(1-\frac{\nu_2}{2}\Big)\,\Gamma\Big(\frac{1}{2}(\nu_1+\nu_2-2)\Big)\ {}_1F_1\Big(1-\frac{\nu_2}{2};\frac{-\nu_1-\nu_2+4}{2};\frac{(c_1+c_2)\,(\tau_1-\tau_2-q)}{2c_1\,c_2}\Big)\Bigg)
\end{aligned} \tag{3.6}$$

for $q < \tau_1 - \tau_2$, $\nu_1 > 0$, $\nu_2 > 0$ and $(1/c_1 + 1/c_2) > 0$; and

$$\begin{aligned}
h_P(q) &= \int_{q+\tau_2}^{\infty} f_{Q_1}(y)\, f_{Q_2}(y-q)\,\mathrm{d}y \\
&= \int_{q+\tau_2}^{\infty} \frac{(y-\tau_1)^{\nu_1/2-1}\,(y-q-\tau_2)^{\nu_2/2-1}\, e^{-(y-\tau_1)/(2c_1)}\, e^{-(y-q-\tau_2)/(2c_2)}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,(2c_1)^{\nu_1/2}\,(2c_2)^{\nu_2/2}}\,\mathrm{d}y \\
&= \frac{(c_1)^{-\nu_1/2}\,(c_2)^{-\nu_2/2}\, e^{(-\tau_1+\tau_2+q)/(2c_1)}}{(-\tau_1+\tau_2+q)\,\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})} \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(2^{(-\nu_1-\nu_2)/2}\,\Gamma\Big(\frac{\nu_2}{2}\Big) \\
&\quad \times \Gamma\Big(-\frac{\nu_1}{2}-\frac{\nu_2}{2}+1\Big)\,{}_1F_1\Big(\frac{\nu_2}{2};\frac{\nu_1+\nu_2}{2};\frac{(c_1+c_2)\,(-\tau_1+\tau_2+q)}{2c_1\,c_2}\Big) \\
&\quad \times \Big(\frac{(c_1+c_2)\,(-\tau_1+\tau_2+q)}{c_1\,c_2}\Big)^{(\nu_1+\nu_2)/2} + \frac{1}{2c_1\,c_2}\,(c_1+c_2)\,(-\tau_1+\tau_2+q) \\
&\quad \times \Gamma\Big(1-\frac{\nu_1}{2}\Big)\,\Gamma\Big(\frac{1}{2}(\nu_1+\nu_2-2)\Big) \\
&\quad \times {}_1F_1\Big(1-\frac{\nu_1}{2};\frac{-\nu_1-\nu_2+4}{2};\frac{(c_1+c_2)\,(-\tau_1+\tau_2+q)}{2c_1\,c_2}\Big)\Big)
\end{aligned} \tag{3.7}$$

for $q \geq \tau_1 - \tau_2$, $\nu_1 > 0$, $\nu_2 > 0$ and $(1/c_1 + 1/c_2) > 0$ where ${}_1F_1(a, b, z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b)\,z^k}{\Gamma(a)\,\Gamma(b+k)\,k!}$. When $q < \tau_1 - \tau_2$, the approximate cumulative distribution function of Q denoted by $F_N(y)$ is then given by

$$\begin{aligned}
F_N(y) &= \int_{-\infty}^{y} h_N(q)\,\mathrm{d}q \\
&= \frac{c_1^{-\nu_1/2}\,c_2^{-\nu_2/2}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_2}{2})\,\Gamma(\frac{\nu_2}{2})} \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(2^{(-\nu_1-\nu_2)/2}\,\Gamma\Big(\frac{\nu_1}{2}\Big)\,\Gamma\Big(1-\frac{\nu_2}{2}-\frac{\nu_1}{2}\Big) \\
&\quad \times \int_{-\infty}^{y} {}_1F_1\Big(\frac{\nu_1}{2};\frac{(\nu_1+\nu_2)}{2};\frac{(c_1+c_2)(\tau_1-\tau_2-q)}{2\,c_1\,c_2}\Big)\, e^{-(\tau_1-\tau_2-q)/(2\,c_2)} \\
&\quad \times \Big(\frac{(c_1+c_2)(\tau_1-\tau_2-q)}{c_1\,c_2}\Big)^{(\nu_1+\nu_2)/2} \frac{1}{(\tau_1-\tau_2-q)}\,\mathrm{d}q + \frac{1}{2\,c_1\,c_2}\,(c_1+c_2) \\
&\quad \times \Gamma\Big(1-\frac{\nu_2}{2}\Big)\;\Gamma\Big(\frac{\nu_1+\nu_2-2}{2}\Big) \int_{-\infty}^{y} e^{-(\tau_1-\tau_2-q)/(2\,c_2)} \\
&\quad \times {}_1F_1\Big(1-\frac{\nu_2}{2};\frac{(-\nu_1-\nu_2+4)}{2};\frac{(c_1+c_2)(\tau_1-\tau_2-q)}{2\,c_1\,c_2}\Big)\,\mathrm{d}q\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{c_1^{-\nu_1/2}\, c_2^{-\nu_2/2}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_2}{2})\,\Gamma(\frac{\nu_2}{2})} \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(\frac{2^{-k-1}\,(c_1+c_2)}{k!\,c_1\,c_2\,\Gamma(k+\frac{1}{2}(-\nu_1-\nu_2+4))} \\
&\quad \times \Big(\frac{c_1+c_2}{c_1\,c_2}\Big)^k \Gamma\Big(k-\frac{\nu_2}{2}+1\Big)\,\Gamma\Big(\frac{\nu_1+\nu_2-2}{2}\Big)\,\Gamma\Big(\frac{-\nu_1-\nu_2+4}{2}\Big) \\
&\quad \times \int_{-\infty}^{y} (\tau_1-\tau_2-q)^k\, e^{(-\tau_1+\tau_2+q)/(2\,c_2)}\,\mathrm{d}q \;+\; \frac{2^{(-\nu_1-\nu_2)/2-k}}{k!\,\Gamma(k+\frac{\nu_1+\nu_2}{2})}\,\Gamma\Big(k+\frac{\nu_1}{2}\Big)\,\Gamma\Big(\frac{\nu_1+\nu_2}{2}\Big) \\
&\quad \times \Gamma\Big(1-\frac{\nu_1+\nu_2}{2}\Big)\,\Big(\frac{c_1+c_2}{c_1\,c_2}\Big)^{(\nu_1+\nu_2-4)/2} \\
&\quad \times \int_{-\infty}^{y} (\tau_1-\tau_2-q)^{k+(\nu_1+\nu_2)/2-1}\, e^{(-\tau_1+\tau_2+q)/(2\,c_2)}\,\mathrm{d}q\Big) \\
&= \sum_{k=0}^{\infty} \frac{2^{-(\nu_1+\nu_2+2)/2}\, c_1^{-\nu_1/2-1}\, c_2^{k-\nu_2/2-1}}{k!\,\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_2}{2})\,\Gamma(\frac{\nu_2}{2})\,\Gamma(k+(\nu_1+\nu_2-2)/2)\,\Gamma(\frac{\nu_1+\nu_2}{2}+k)} \\
&\quad \times \Big(\frac{1}{c_1}+\frac{1}{c_2}\Big)^k \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(4\,c_1\,c_2^2\,\Gamma\Big(\frac{\nu_1+\nu_2}{2}\Big)\,\Gamma\Big(k+\frac{\nu_1}{2}\Big) \\
&\quad \times \Gamma\Big(k+\frac{\nu_1+\nu_2-2}{2}\Big)\,\Gamma\Big(k+1,-\frac{z-\tau_1+\tau_2}{2\,c_2}\Big)\,\Big(\frac{1}{c_1}+\frac{1}{c_2}\Big)^{(\nu_1+\nu_2)/2} \\
&\quad + 2^{\nu_1+\nu_2}\, c_2^{(\nu_1+\nu_2)/2}\,(c_1+c_2)\,\Gamma\Big(-\frac{\nu_1+\nu_2-4}{2}\Big)\,\Gamma\Big(k+1-\frac{\nu_2}{2}\Big)\,\Gamma\Big(\frac{1}{2}(\nu_1+\nu_2-2)\Big) \\
&\quad \times \Gamma\Big(k+\frac{\nu_1+\nu_2}{2}\Big)\,\Gamma\Big(\frac{1}{2}(2\,k+\nu_1+\nu_2),-\frac{z-\tau_1+\tau_2}{2\,c_2}\Big)\Big)\,, \qquad (3.8)
\end{aligned}
$$

where $\Gamma(\alpha,z)=\int_z^{\infty} x^{\alpha-1}\,e^{-x}\,\mathrm{d}x$ denotes the incomplete gamma function. Similarly, when $q \geq \tau_1-\tau_2$, the approximate cumulative distribution function of Q denoted by $F_P(y)$ can be expressed as

$$
\begin{aligned}
F_P(y) &= F_N(\tau_1-\tau_2) + \int_{\tau_1-\tau_2}^{y} h_P(q)\,\mathrm{d}q \\
&= F_N(\tau_1-\tau_2) \\
&\quad + \sum_{k=0}^{\infty} \frac{c_1^{-\nu_1/2}\, c_2^{-\nu_2/2}}{\Gamma(\frac{\nu_1}{2})\,\Gamma(1-\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})} \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(\frac{2^{-k-1}\,(c_1+c_2)}{k!\,c_1\,c_2\,\Gamma(k+\frac{1}{2}(-\nu_1-\nu_2+4))} \\
&\quad \times \Big(\frac{c_1+c_2}{c_1\,c_2}\Big)^k \Gamma\Big(k-\frac{\nu_1}{2}+1\Big)\,\Gamma\Big(\frac{-\nu_1-\nu_2+4}{2}\Big)\,\Gamma\Big(\frac{\nu_1+\nu_2-2}{2}\Big) \\
&\quad \times \int_{\tau_1-\tau_2}^{y} (-\tau_1+\tau_2+q)^k\, e^{-(-\tau_1+\tau_2+q)/(2\,c_1)}\,\mathrm{d}q \;+\; \frac{2^{(-\nu_1-\nu_2)/2-k}}{k!\,\Gamma(k+\frac{\nu_1+\nu_2}{2})} \\
&\quad \times \Gamma\Big(\frac{-\nu_1}{2}-\frac{\nu_2}{2}+1\Big)\,\Gamma\Big(k+\frac{\nu_2}{2}\Big)\,\Gamma\Big(\frac{\nu_1+\nu_2}{2}\Big)\,\Big(\frac{c_1+c_2}{c_1\,c_2}\Big)^{k+(\nu_1+\nu_2)/2} \\
&\quad \times \int_{\tau_1-\tau_2}^{y} (-\tau_1+\tau_2+q)^{k+(\nu_1+\nu_2)/2-1}\, e^{-(-\tau_1+\tau_2+q)/(2\,c_1)}\,\mathrm{d}q\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= F_N(\tau_1-\tau_2) \\
&+ \sum_{k=0}^{\infty} \frac{c_1^{k-\nu_1/2}\, c_2^{-1-\nu_2/2}}{k!\,\Gamma(1-\frac{\nu_1}{2})\,\Gamma(\frac{\nu_1}{2})\,\Gamma(\frac{\nu_2}{2})\,\Gamma(k+2-\frac{\nu_1+\nu_2}{2})\,\Gamma(\frac{\nu_1+\nu_2}{2}+k)} \\
&\times \Big(\frac{1}{c_1}+\frac{1}{c_2}\Big)^k \Big(\frac{c_1\,c_2}{c_1+c_2}\Big)^{(\nu_1+\nu_2)/2} \Big(c_1^{(\nu_1+\nu_2)/2}\, c_2\,\Gamma\Big(1-\frac{\nu_1}{2}-\frac{\nu_2}{2}\Big) \\
&\times \Gamma\Big(k+2-\frac{\nu_1+\nu_2}{2}\Big)\,\Gamma\Big(k+\frac{\nu_2}{2}\Big)\,\Gamma\Big(\frac{\nu_1+\nu_2}{2}\Big)\,\Big(\Gamma\Big(\frac{1}{2}\,(2\,k+\nu_1+\nu_2)\Big) \\
&- \Gamma\Big(\frac{1}{2}\,(2\,k+\nu_1+\nu_2), \frac{z-\tau_1+\tau_2}{2\,c_1}\Big)\Big)\,\Big(\frac{1}{c_1}+\frac{1}{c_2}\Big)^{(\nu_1+\nu_2)/2} + (c_1+c_2) \\
&\times \Gamma\Big(k+1-\frac{\nu_1}{2}\Big)\,\Gamma\Big(\frac{1}{2}\,(-\nu_1-\nu_2-4)\Big)\,\Gamma\Big(\frac{1}{2}\,(\nu_1+\nu_2-2)\Big)\,\Gamma\Big(k+\frac{\nu_1+\nu_2}{2}\Big) \\
&\times \Big(\Gamma(k+1)-\Gamma\Big(k+1,\frac{z-\tau_1+\tau_2}{2\,c_1}\Big)\Big)\Big)\,. \qquad (3.9)
\end{aligned}
$$

It was observed that the infinite sums involved in the representations of the cumulative distribution function approximants can be truncated to fifty terms for computational purposes. It should also be noted that polynomially-adjusted Pearson-type approximants, which can be determined by making use of the technique described in Section 3.3, generally provide more accurate approximations.

3.2 Approximations by Means of Gamma-Type Distributions

It is explained in this section that gamma-type approximations can also be used to approximate the distribution of a noncentral quadratic form. The density function of the two-parameter gamma distribution is given by

$$
\psi(x) \;=\; \frac{x^{\alpha-1}\,e^{-x/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}\,, \quad x>0\;,\;\alpha>0 \text{ and } \beta>0\,. \qquad (3.10)
$$

The parameters α and β can be estimated as follows on the basis of its first two raw moments denoted $\mu(1)$ and $\mu(2)$: $\alpha=\mu(1)^2/(\mu(2)-\mu(1)^2)$ and $\beta=\mu(2)/\mu(1)-\mu(1)$.

A three-parameter gamma or generalized gamma density function is given by

$$
\psi(x)=\frac{\gamma}{\beta^{\alpha\,\gamma}\,\Gamma(\alpha)}\,x^{\alpha\,\gamma-1}e^{-(x/\beta)^{\gamma}}\,\mathcal{I}_{(0,\infty)}(x)\,,\;\alpha>0,\;\beta>0,\text{ and }\gamma>0, \qquad (3.11)
$$

where $\alpha>0,\;\gamma>0$ and $\beta>0$. Denoting its moments by $m(j),\; j=0,1,\ldots,$ one has,

$$
m(j)=\frac{\beta^{j}\,\Gamma(\alpha+j/\gamma)}{\Gamma(\alpha)}\,. \qquad (3.12)
$$

Given the first three moments of the positive quadratic form Q_1, the three parameters of the generalized gamma distribution can readily be estimated by making use of the method of moments. The estimates are thus obtained by solving simultaneously the equations

$$
\mu_{Q_1}(i) \;=\; m(i)\,, \quad \text{for } i=1,2,3, \qquad (3.13)
$$

which are nonlinear. We proceed similarly for approximating the density function of Q_2 as defined in (2.3). Then (2.6) yields the approximate density function of $Q_1 - Q_2$ with $\tau_1 = \tau_2 = 0$. Clearly, the gamma distribution is a particular case of the generalized gamma distribution wherein the parameter γ equal one.

3.3 Polynomially-Adjusted Density Approximants

A density approximation technique that is based on the first n moments of an indefinite quadratic form is being proposed in this section. In order to approximate the density function of a noncentral quadratic form Q, one must first approximate the density functions of the two positive definite quadratic forms Q_1 and Q_2, as defined in (2.3). According to Equation (2.5), the moments of the positive definite quadratic form Q_1 denoted by $\mu_{Q_1}(\cdot)$ are obtained recursively from its cumulants. Then, a moment-based density approximation of the following form is assumed for Q_1:

$$f_n(x) = \psi(x) \sum_{j=0}^{n} \xi_j \, x^j , \tag{3.14}$$

where $\psi(x)$ is an initial density approximant also referred to as base density function, which could be for instance a gamma, generalized gamma or Pearson-type density function.

In order to determine the polynomial coefficients, ξ_j, we equate the h^{th} moment of Q_1 to the h^{th} moment of the approximate distribution specified by $f_n(x)$. That is, we let

$$\begin{aligned} \mu_{Q_1}(h) &= \int_{\tau_1}^{\infty} x^h \psi(x) \sum_{j=0}^{n} \xi_j \, x^j \, \mathrm{d}x \\ &= \sum_{j=0}^{n} \xi_j \int_{\tau_1}^{\infty} x^{h+j} \, \psi(x) \, \mathrm{d}x \\ &= \sum_{j=0}^{n} \xi_j \; m(h+j), \quad \text{for } h = 0, 1, \ldots, n, \end{aligned} \tag{3.15}$$

where $m(h+j)$ denotes the $(h+j)^{\text{th}}$ moment determined from $\psi(x)$ and τ_1 is the lower bound of the support of $\psi(x)$. For the gamma and the generalized gamma, $\tau_1 = 0$ and $m(j)$ is given by (3.12) while in the case of the Pearson-type distribution,

$$m(j) = \begin{cases} \frac{2^{-\nu/2}\, c^h \, e^{\tau/(2\,c)}}{\Gamma(1-\frac{\nu}{2})\,\Gamma(h+1+\frac{\nu}{2})\,\Gamma(\frac{\nu}{2})} \Big(\Gamma(h+1) \Big(\Gamma(-h-\frac{\nu}{2})\,\Gamma(h+\frac{\nu}{2}+1)\,(-\frac{\tau}{c})^h + (\frac{\tau}{c})^h\,\Gamma(1-\frac{\nu}{2}) \\ \quad \times \; \Gamma(\frac{\nu}{2}) \Big) \, {}_1F_1\Big(h+1; h+\frac{\nu}{2}+1; -\frac{\tau}{2c}\Big)\,(-\frac{\tau}{c})^{\nu/2} + 2^{h+\nu/2}\,\Gamma(1-\frac{\nu}{2})\,\Gamma(h+\frac{\nu}{2}) \\ \quad \times \; \Gamma(h+\frac{\nu}{2}+1)\,{}_1F_1\Big(1-\frac{\nu}{2}; \frac{1}{2}(-2\,h-\nu+2); -\frac{\tau}{2c}\Big)\Big) \quad \text{for } \tau \le 0\,, \\ 2^h\, c^h \, U\Big(-h, 1-h-\frac{\nu}{2}, \frac{\tau}{2c}\Big) \quad \text{for } \tau > 0\,, \end{cases}$$

where $U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-z\,t}\, t^{a-1}\,(1+t)^{b-a-1} \mathrm{d}t$ is the confluent hypergeometric function. This leads to a linear system of $(n+1)$ equations in $(n+1)$ unknowns whose solution is

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} m(0) & m(1) & \cdots & m(n-1) & m(n) \\ m(1) & m(2) & \cdots & m(n) & m(n+1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m(n) & m(n+1) & \cdots & m(2n-1) & m(2n) \end{pmatrix}^{-1} \begin{pmatrix} \mu_{Q_1}(0) \\ \mu_{Q_1}(1) \\ \vdots \\ \mu_{Q_1}(n) \end{pmatrix} . \quad (3.16)$$

The resulting representation of the density function of Q_1 will be referred to as a polynomially-adjusted density approximant, which can be readily evaluated. As long as higher moments are available, more accurate approximations can always be obtained by making use of additional exact moments.

The same procedure will produce a polynomially-adjusted density approximant for Q_2. The density approximant to the noncentral indefinite quadratic form $Q = Q_1 - Q_2$ is then obtained from Equation (2.6). Note that in the case of gamma-type base density functions, one should set τ_1 and τ_2 equal to zero in (2.6), (2.7), (2.8) and (3.15).

4 The Algorithm

The following algorithm can be utilized to approximate the density function of the quadratic form $Q = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$, and A is a symmetric indefinite matrix.

1. The eigenvalues of $A\Sigma$ denoted by $\lambda_1 \geq \cdots \geq \lambda_r > 0 > \lambda_{r+\theta+1} \geq \cdots \geq \lambda_p$, and the *corresponding normalized* eigenvectors, $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p$, are determined.

2. Letting $P = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_p)$, $\gamma_1, \ldots, \gamma_p$ be the eigenvalues of Σ, $\mathbf{t}_1, \ldots, \mathbf{t}_p$ be the *normalized* eigenvectors corresponding to $\gamma_1, \ldots, \gamma_p$, $T = (\mathbf{t}_1, \ldots, \mathbf{t}_p)$,
$$\Sigma^{-1/2} = T\,\mathrm{Diag}(\gamma_1^{-1/2}, \ldots, \gamma_p^{-1/2})T',$$
$\boldsymbol{b} = (b_1, \cdots, b_p)' = P'\,\Sigma^{-1/2}\,\boldsymbol{\mu}$ and $U_1, \ldots, U_p$ be independently distributed standard normal variables, one has the decomposition
$$Q = \sum_{j=1}^{r} \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^{p} |\lambda_j| (U_j + b_j)^2 \equiv Q_1 - Q_2,$$
where $Q_1 \equiv \mathbf{W}_1' A_1 \mathbf{W}_1$, $\mathbf{W}_1 \sim \mathcal{N}_r(\boldsymbol{b_1}, I)$, $\boldsymbol{b_1} = (b_1, \ldots, b_r)'$, $A_1 = \mathrm{Diag}(\lambda_1, \ldots, \lambda_r)$, and $Q_2 \equiv \mathbf{W}_2' A_2 \mathbf{W}_2$, $\mathbf{W}_2 \sim \mathcal{N}_{p-r-\theta}(\boldsymbol{b_2}, I)$, $\boldsymbol{b_2} = (b_{r+\theta+1}, \ldots, b_p)'$, $A_2 = \mathrm{Diag}(|\lambda_{r+\theta+1}|, \ldots, |\lambda_p|)$. Clearly, $\mathbf{b} = \mathbf{0}$ whenever $\boldsymbol{\mu} = \mathbf{0}$ and, in that case, there is no need to determine the matrices P or T.

3. The cumulants and the moments of Q_1 and Q_2 are determined from Equations (2.4) and (2.5), respectively.

4. Density approximants are obtained for each of the positive definite quadratic forms Q_1 and Q_2 on the basis of their respective moments and denoted by $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$.

5. Given $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, the approximate density of Q is determined from Equation (2.6) where $h_P(\cdot)$ and $h_N(\cdot)$ are respectively specified by Equations (2.7) and (2.8). In the case of Pearson-type approximants, $h_P(\cdot)$ and $h_N(\cdot)$ are explicitly given by (3.6) and (3.7). When explicit representations of $h_P(\cdot)$ and $h_N(\cdot)$ are unavailable, numerical integration can be used.

6. As explained in Section 3.3, the accuracy of the approximants of Q_1 and Q_2 can be improved upon by making use of polynomial adjustments. Then again, (2.6) can be used to obtain an approximate density function for Q.

7. The cumulative distribution function of Q can then be evaluated from Equations (3.8) and (3.9) in the case of Pearson's approximation and by numerical integration in other cases.

Remark 1. For the nonnegative definite quadratic form, $Q = \mathbf{X}'A\mathbf{X}$, $A \geq 0$, whose eigenvalues are all nonnegative, only the distribution of Q_1 needs be approximated.

5 Numerical Examples

We present two numerical examples in this section. The first involves a noncentral indefinite quadratic form while the second approximates the distribution of the Durbin-Watson statistic, which is shown to be expressible as a ratio of quadratic forms.

Example 1

Consider the noncentral indefinite quadratic form, $Q = \mathbf{X}'A\mathbf{X}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 2 & 8 & 0 & 4 \\ 2 & 0 & -1/4 & 1 \\ 5 & 4 & 1 & -2 \end{pmatrix}$$

and $\mathbf{X} \sim \mathcal{N}_4(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} = (1, 2, 3, 4)'$ and

$$\Sigma = \begin{pmatrix} 1 & -1/2 & 2/5 & 1/2 \\ -1/2 & 1 & 1/4 & -3/8 \\ 2/5 & 1/4 & 1 & 1/3 \\ 1/2 & -3/8 & 1/3 & 1 \end{pmatrix}.$$

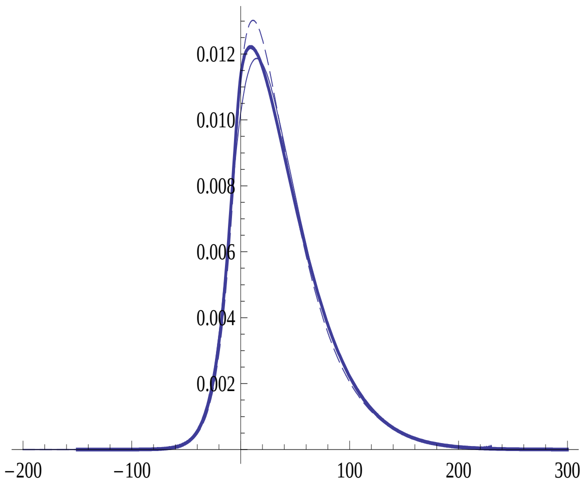

Figure 1: *Three Density Approximants: Pearson's (light solid line), Generalized Gamma (dark solid line) and Gamma (dashed line)*

In light of Equation (2.3), Q can be re-expressed as

$$Q = Q_1 - Q_2 = \sum_{i=1}^{2} \lambda_i (U_i + b_i)^2 - \sum_{j=3}^{4} |\lambda_j| (U_j + b_j)^2 \,, \tag{5.1}$$

where the U_i's, $i = 1, 2, 3, 4$, are standard normal random variables, $\lambda_1 = 8.29749$, $\lambda_2 = 4.61802$, $\lambda_3 = -3.25405$, $\lambda_4 = -0.644806$, $b_1 = 2.13221$, $b_2 = 0.519464$, $b_3 = -1.67346$, and $b_4 = -2.52353$. In this case, the matrices $\Sigma^{1/2}$ and P are respectively

$$\Sigma^{1/2} = \begin{pmatrix} 0.909305 & -0.272122 & 0.222592 & 0.222637 \\ -0.272122 & 0.926505 & 0.182801 & -0.184722 \\ 0.222592 & 0.182801 & 0.942687 & 0.168459 \\ 0.222637 & -0.184722 & 0.168459 & 0.942301 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0.593908 & 0.3517 & 0.53923 & 0.482506 \\ -0.399612 & 0.90875 & -0.111034 & -0.0464278 \\ 0.472832 & 0.179678 & 0.125688 & -0.853433 \\ 0.513822 & 0.134896 & -0.825291 & 0.191533 \end{pmatrix}.$$

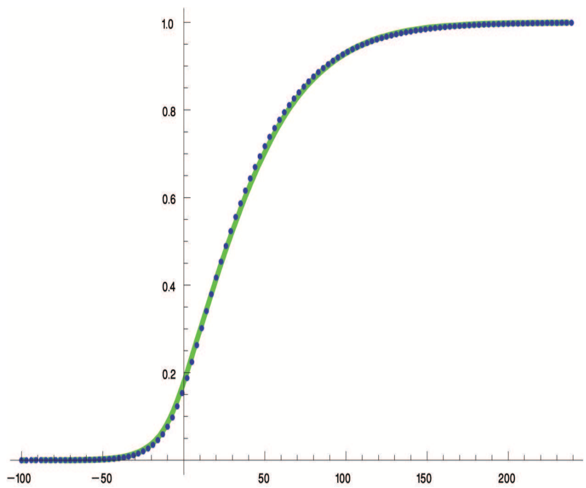

Figure 2: *Gamma cdf approximation (dotted curve) and simulated cdf*

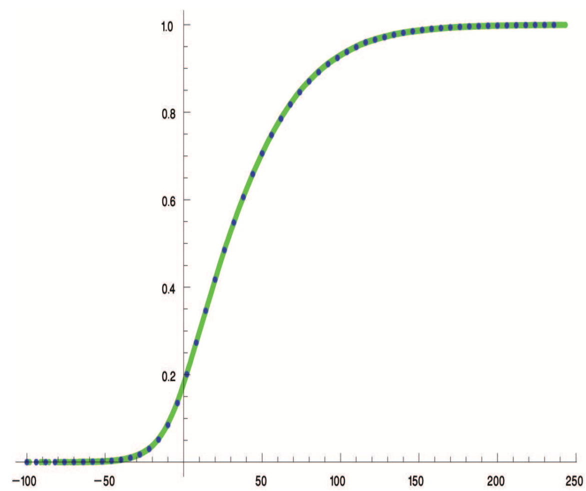

Figure 3: *Generalized gamma cdf approximation (dotted curve) and simulated cdf*

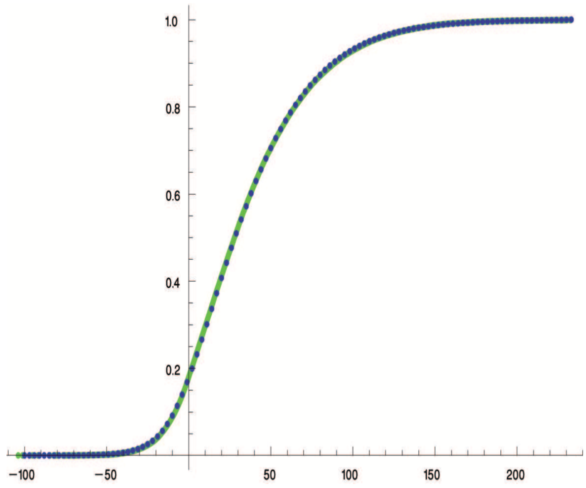

Figure 4: *Pearson's cdf approximation (dotted curve) and simulated cdf*

The approximate density functions of Q_1 and Q_2 were obtained by making use of the gamma, generalized gamma and Pearson's density functions. The resulting approximations of the density function of Q, as evaluated from Steps 4 and 5 of the proposed algorithm, are plotted in Figure 1. The corresponding cumulative distribution functions which were determined by making use of the last step of the algorithm, are respectively plotted in Figures 2, 3 and 4 where they are superimposed on the simulated distribution function which was generated from 100,000 replications. It is apparent that Pearson's approximation is the most accurate.

Example 2

The statistic originally proposed by Durbin and Watson (1950), which assesses whether the disturbances in the linear regression model

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{5.2}$$

are uncorrelated, can be expressed as

$$D = \frac{\hat{\boldsymbol{\epsilon}}' A^* \hat{\boldsymbol{\epsilon}}}{\hat{\boldsymbol{\epsilon}}' \hat{\boldsymbol{\epsilon}}}\,, \tag{5.3}$$

where

$$\hat{\boldsymbol{\epsilon}} = \mathbf{Y} - X\hat{\boldsymbol{\beta}} \tag{5.4}$$

is the vector of residuals,

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\mathbf{Y} \tag{5.5}$$

Table 1: Approximate cdf's evaluated from (5.8) at certain simulated percentiles

CDF	Simulated	P-A Gamma	P-A Gen. Gamma	P-A Pearson
0.01	1.36069	0.010435	0.010420	0.010197
0.05	1.64792	0.050280	0.050277	0.050286
0.1	1.80977	0.099761	0.099770	0.100059
0.2	2.00943	0.198599	0.198625	0.198878
0.25	2.08536	0.247875	0.247909	0.248167
0.4	2.27598	0.396053	0.396086	0.396270
0.5	2.39014	0.495934	0.495953	0.496051
0.6	2.50282	0.596315	0.596317	0.596184
0.75	2.6861	0.748343	0.748288	0.747567
0.8	2.75694	0.799397	0.799323	0.798449
0.9	2.93742	0.902156	0.902100	0.901533
0.95	3.07679	0.952783	0.952788	0.952592
0.99	3.31005	0.991466	0.991457	0.992064

being the ordinary least-squares estimator of $\boldsymbol{\beta}$, and $A^* = (a^*_{ij})$ is a symmetric tridiagonal matrix with $a^*_{11} = a^*_{pp} = 1$; $a^*_{ii} = 2$, for $i = 2, \ldots, p-1$; $a^*_{ij} = -1$ if $|i-j| = 1$; and $a^*_{ij} = 0$ if $|i-j| \geq 2$. Assuming that the error vector is normally distributed, one has $\boldsymbol{\epsilon} \sim \mathcal{N}_p(\mathbf{0}, I)$ under the null hypothesis.

Then, on writing $\hat{\boldsymbol{\epsilon}}$ as $M\mathbf{Y}$ where

$$M_{p\times p} = I - X(X'X)^{-1}X' = M' \tag{5.6}$$

is an idempotent matrix of rank $p-k$, the test statistic can be expressed as the ratio of quadratic forms,

$$D = \frac{\mathbf{Z}'MA^*M\mathbf{Z}}{\mathbf{Z}'M\mathbf{Z}}\,, \tag{5.7}$$

where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$; this can be seen from the fact that $M\mathbf{Y}$ and $M\mathbf{Z}$ are identically distributed singular normal vectors with mean vector $\mathbf{0}$ and covariance matrix MM'. We note that the distribution function of D at t_0 can be determined as follows:

Table 2: Approximate cdf's based directly on the moments of D

CDF	Simulated	Gen. Gamma	P-A Gen. Gamma
0.01	1.36069	0.011744	0.010365
0.05	1.64792	0.050061	0.050308
0.1	1.80977	0.097460	0.099875
0.2	2.00943	0.194079	0.198706
0.25	2.08536	0.243139	0.247947
0.3	2.15357	0.292786	0.297291
0.5	2.39014	0.495703	0.495807
0.7	2.62156	0.702657	0.697458
0.75	2.68610	0.754125	0.748325
0.8	2.75694	0.805216	0.799434
0.9	2.93742	0.905234	0.902239
0.95	3.07679	0.952770	0.952814
0.99	3.31005	0.989273	0.991458

$$\begin{aligned}\Pr(D < t_0) &= \Pr\Big(\mathbf{Z}'MA^*M\mathbf{Z} < t_0\mathbf{Z}'M\mathbf{Z}\Big)\\ &= \Pr\Big(\mathbf{Z}'M(A^*M - t_0 I)\mathbf{Z} < 0\Big). \qquad (5.8)\end{aligned}$$

On letting $U = \mathbf{Z}'M(A^*M - t_0I)\mathbf{Z}$, U can be re-expressed as $Q_1 - Q_2$, the difference of two positive quadratic forms, by applying Steps 1 and 2 of the algorithm described in Section 3, with $A = M(A^*M - t_0I)$, $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$. Polynomially-adjusted density approximants of degree 10 were then obtained by applying Steps 3 and 6 of the methodology.

We make use of a data set that is provided in Hildreth and Lu (1960, p. 58). In this case, there are $k = 5$ independent variables, $p = 18$, the observed value of D is 0.96, and the 13 non-zero eigenvalues of $M(A^*M - t_0I)$ are those of MA^*M minus t_0. The non-zero eigenvalues of MA^*M are 3.92807, 3.82025, 3.68089, 3.38335, 3.22043, 2.95724, 2.35303, 2.25696, 1.79483, 1.48804, 0.948635, 0.742294 and 0.378736. For instance, when $t_0 = 1.8099$, which corresponds to the 10^{th} percentile of the simulated cumulative distribution function resulting from 1,000,000 replications, the eigenvalues of the positive definite quadratic form Q_1 are 2.11817, 2.01035, 1.87099, 1.57345, 1.41053, 1.14734, 0.54313 and 0.44706, while those of Q_2 are 0.01507, 0.32186, 0.861265, 1.06761 and 1.43116. The density function approximations of D were obtained on the basis of gamma, generalized gamma and Pearson-type initial density functions. The corresponding polynomially-adjusted cumulative distribution functions were evaluated at certain percentiles of the simulated distribution. The results reported in Table 1 indicate that the polynomially-adjusted Pearson-type (P-A Pearson) approximation is generally the most accurate.

Since M, the matrix of quadratic form appearing in the denominator of D as defined in (5.7), is idempotent, we can apply a result stated in Hannan (1970), namely that, in this case, the moments of the ratio of the quadratic forms are equal to the ratio of the moments. Thus, the h^{th} moment of D can be obtained as $E(\mathbf{Z}'MA^*M\mathbf{Z})^h/E(\mathbf{Z}'M\mathbf{Z})^h$ and a polynomially-adjusted generalized gamma density approximant as defined in Section (3.3) can be directly determined from the exact moments of D.

The approximate cumulative distribution function for the generalized gamma (Gen. Gamma) as well as the tenth-degree polynomially-adjusted generalized gamma (P-A Gen. Gamma) were evaluated at certain percentiles obtained from the distribution of the ratio, which was generated from 1,000,000 replications. The results reported in Table 2 indicate that the proposed approximations are indeed very accurate. All the calculations were carried out with the symbolic computational package *Mathematica*, the code being available from the authors upon request.

Acknowledgements

This research was supported by the Natural Sciences and Engineering Research Council of Canada. We would also like to acknowledge with thanks Hyung-Tae Ha's initial contributions to the development of the results.

References

[1] Anderson, O. D. (1990). Moments of the sampled autocovariances and autocorrelations for a Gaussian white-noise process. *Canad. J. Statist.*, **18**, 271–284.

[2] Box, G. E. P. (1954). Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification. *Ann. Math. Statist.*, **25**, 290–302.

[3] Davis, R. B. (1973). Numerical inversion of a characteristic function, *Biometrika*, **60**, 415–417.

[4] De Gooijer, J. G. and MacNeill, I. B. (1999). Lagged regression residuals and serial correlation tests. *Journal of Business and Economomic Statistics*, **17**, 236–247.

[5] Dempster, A. P., Schatzoff, M. and Wermouth, N. (1977). A simulation study of alternatives to ordinary least squares. *Journal of the American Statistical Association*, **72**, 77–106.

[6] Durbin, J. and Watson, G.S. (1950). Testing for serial correlation in least squares regression, I. *Biometrika*, **37**, 409-428.

[7] Gurland, J. (1953). Distribution of quadratic forms and ratios of quadratic forms. *Ann. Math. Statist.*, **24**, 416–427.

[8] Gurland, J. (1956). Quadratic forms in normally distributed random variables. *Sankhyā*, **17**, 37–50.

[9] Hannan, E. J.(1970). *Multiple Time Series.* John Wiley and Sons, New York.

[10] Hendry, D. F.(1979). The behaviour of inconsistent instrumental variables estimators in dynamic systems with autocorrelated errors. *Journal of Econometrics*, **9**, 295–314.

[11] Hendry, D. F. and Harrison, R. W. (1974). Monte Carlo methodology and the small sample behaviour of ordinary and two-stage least squares. *Journal of Econometrics*, **2**, 151–174.

[12] Hendry, D. F. and Mizon, G. (1980) An empirical application and Monte Carlo analysis of tests of dynamic specification. *Review of Economic Studies*, **47**, 21–45.

[13] Hildreth, C. and Lu, J. Y. (1960). *Demand Relations with Auto-Correlated Disturbances.* East Lansing, Michigan: Michigan State University, Agricultural Experiment Station, Department of Agricultural Economics, Technical Bulletin 276.

[14] Imhof, J. P. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika*, **48**, 419–426.

[15] Inder, B. (1986). An approximation to the null distribution of the Durbin-Watson test statistic in models containing lag dependent variables. *Econometric Theory*, **2**, 413–428.

[16] Koerts, J. and Abrahamse, A. P. J. (1969). *The Theory and Application of the General Linear Model.* University Press, Rotterdam.

[17] Kotz, S., Johnson, N. L. and Boyd, D. W. (1967a). Series representation of distribution of quadratic forms in normal variables I. Central case. *Ann. Math. Statist.*, **38**, 823–837.

[18] Kotz, S., Johnson, N. L. and Boyd, D. W. (1967b). Series representation of distribution of quadratic forms in normal variables II. Non-central case. *Ann. Math. Statist.*, **38**, 838–848.

[19] MacNeill, I. B. (1978). Properties of sequences of partial sums of polynomial regression residuals with applications to tests for change of regression at unknown times. *The Annals of Statistics*, **6**, 422–433.

[20] Mathai, A. M. and Provost, S. B. (1992). *Quadratic Forms in Random Variables, Theory and Applications.* Marcel Dekker Inc., New York.

[21] McLeod, A. I. (1979). Distribution of the residual cross-correlation in univariate ARMA time series models. *Journal of the American Statistical Association*, **74**, 849–855.

[22] Pachares, J. (1955). Note on the distribution of a definite quadratic form. *Ann. Math. Statist.*, **26**, 128–131.

[23] Pearson, E.S. (1959). Note on an approximation to the distribution of noncentral χ^2. *Biometrika*,**46**, 364.

[24] Provost, S. B. and Rudiuk, E. M. (1995). The sampling distribution of the serial correlation coefficient. *The American Journal of Mathematical and Management Sciences*, **15**(3), 57-81.

[25] Provost, S. B., Sanjel, D. and MacNeill, I. B. (2005). On approximating the distribution of an alternative statistic for detecting lag-k correlation. *Journal of Probability and Statistical Science*, **3(2)**, 229–239.

[26] Rice, S. O. (1980). Distribution of quadratic forms in normal variables. Evaluation by numerical integration. *SIAM J. Scient. Statist. Comput.*, **1**, 438–448.

[27] Ruben, H. (1960). Probability content of regions under spherical normal distribution. I. *Ann. Math. Statist.*, **31**, 598–619.

[28] Ruben, H. (1962). Probability content of regions under spherical normal distribution. IV: The distribution of homogeneous and nonhomogeneous quadratic functions in normal variables. *Ann. Math. Statist.*, **33**, 542–570.

[29] Shah, B. K. (1963). Distribution of definite and of indefinite quadratic forms from a non-central normal distribution. *Ann. Math. Statist.*, **34**, 186–190.

[30] Shah, B. K. and Khatri, C. G. (1961). Distribution of a definite quadratic form for non-central normal variates. *Ann. Math. Statist.*, **32**, 883–887. (Corrections **34**, 673).

[31] Shenton, L. R. and Johnson, W. L. (1965). Moments of a serial correlation coefficient. *J. Roy. Statist. Soc., Ser. B*, **27**, 308–320.

[32] Smith, P. J. (1995). A recursive formulation of the old problem of obtaining moments from cumulants and vice versa. *The American Statistician*, **49**, 217–219.