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MATHEMATICAL METHODS OF CONSTRUCTING GENERALIZATIONS OF SKEW NORMAL DISTRIBUTIONS

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SUMMARY

In this paper, two different mathematical methods are used to derive skew distributions. The results generalize Azzalini (1985) and Fernández and Steel (1998) skew distributions generating new results in general, and skew normal distribution in particular. Some mathematical properties, such as n-th moments, distribution function, moments generating function, are also given for the generalizations. Some known and new special cases are also mentioned. Some graphs for skew distributions are included. Results are applied to two practical problems. Shannon and Renyi entropies as well as Fisher information are also obtained.

Keywords and phrases: Generalization of skew normal distribution, Generalization of Fernández and Steel distribution, Information theory

AMS Classification: 60E05, 94A17

1 Introduction

Skew symmetric models have been considered by several researchers. Skew normal distribution is a classical example. Skew distributions are useful is many practical situations (see, for example Rathie et al., 2008). For completeness of this paper, a definition is given in this section. The probability density function of standard normal distribution will be denoted by $\phi(x)$ and its distribution function by $\Phi(x)$. The error function (Kotz et al., 1994) erf$[x]$ is defined as (Maclaurin series)

$$\text{erf}[z] \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (z)^{2n+1}}{n!(2n+1)}. \tag{1.1}$$

In this paper we generalized Azzalini skew distribution and obtained the moment generating function, and distribution of X^2 in Section 2. Two applications, with real data, are given in Section 2.3. Fernández and Steel skew distribution is generalized in Section 3 and some properties, including Shannon entropy, Renyi entropy and Fisher information, are given. Results for asymmetric normal distribution are also given. Several graphs are drawn to demonstrate the applicability of the distributions.

2 Generalization of Azzalini skew distributions

Azzalini (1985) obtained the following density function for skew distribution

$$h(x) = 2f(x)G(w(x)), \quad x \in (-\infty, \infty), \tag{2.1}$$

where $f(x)$ is symmetric (about the origin) probability density function of a random variable X, $G(x)$ is any cumulative distribution function of another symmetric (about the origin) density function g(x), and $w(x)$ is an odd function of x.

For $f(x) = \phi(x)$ and $G(cx) = \Phi(cx)$, we obtain the well known skew normal density function (Azzalini, 1985)) given by

$$h(x) = 2\phi(x)\Phi(cx), \quad c, x \in (-\infty, \infty). \tag{2.2}$$

Generalizing with a position and a scale parameter, μ and σ, we get

$$Sk(x) = \frac{2}{\sigma}\phi(\frac{x-\mu}{\sigma})\Phi(c\frac{x-\mu}{\sigma}), \quad c, x \in (-\infty, \infty). \tag{2.3}$$

The density function (2.2) with $c = \rho(1-\rho^2)^{-.5}$, can also be obtained from a bivariate normal density function. Let (X, Y) be distributed as $N_2(\mu, \mathbf{\Sigma})$ with mean vector $\mu = (0, 0)^T$ and variance-covariance matrix $\Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}$, where ρ is the correlation coefficient. It is easy to prove that

$$h(x) = P\{X|Y > 0\} = 2\phi(x)\Phi\left(\frac{\rho}{\sqrt{1-\rho^2}}x\right). \tag{2.4}$$

Thus we can use the bivariate normal distribution to create a new class of Skew distributions generalizing (2.4) by taking $a, b \in R$, $a < b$, such that,

$$\begin{aligned} h(x) = P\{X|a < Y < b\} &= \frac{P(\{a < Y < b\}, X)}{P(\{a < Y < b\})} \\ &= \frac{\phi(x)}{P(\{a < Y < b\})}\int_a^b \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2\right\}dy \end{aligned}$$

and using the substitution $z = (y-\rho x)/\sqrt{1-\rho^2}$, we get

$$h(x) = \phi(x)\left\{\Phi\left[\frac{b-\rho x}{\sqrt{1-\rho^2}}\right] - \Phi\left[\frac{a-\rho x}{\sqrt{1-\rho^2}}\right]\right\}\{\Phi(b) - \Phi(a)\}^{-1}. \tag{2.5}$$

Clearly, for $a = 0$ and $b \to \infty$, (2.5) reduces to (2.4). Similarly, for $b = 0$ and $a \to -\infty$ (2.5) reduces to (2.4) with ρ replaced by $-\rho$.

The moment generating function for (2.5) is given by

$$M_X[t] = \frac{\exp\{t^2/2\}}{2(\Phi(b) - \Phi(a))}\left[\text{erf}\left[\frac{b-\rho t}{\sqrt{2}}\right] - \text{erf}\left[\frac{a-\rho t}{\sqrt{2}}\right]\right]. \tag{2.6}$$

Or equivalently:

$$M_X[t] = \exp\{t^2/2\}\frac{\Phi(b-\rho t)-\Phi(a-\rho t)}{\Phi(b)-\Phi(a)}. \tag{2.7}$$

where the error function is defined in (1.1).

The density function $h(x)$ in (2.5) is plotted for two sets of values of a,b and ρ in Figure 1, and compared with the N(0,1) distribution.

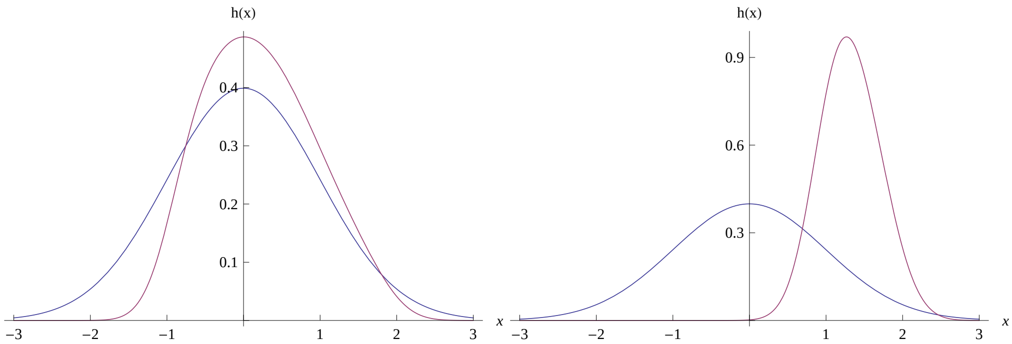

Figure 1: *Skew-normal with* $a=-1$, $b=2$ *and* $\rho=0.95$ *(left) and Skew-normal with* $a=1$, $b=2$ *and* $\rho=0.95$ *(right). Both compared with* $N(0,1)$ *distribution.*

2.1 $X^2_{(a,b)}$ Distribution: generalization of chi-square distribution

It is well known that if X is a random variable with skew normal distribution, then $Y = X^2$ is a random variable with Chi-Square distribution (Lin et al. (2004)). So, let X be a random variable with distribution given by (2.5), then $Z = X^2$ is a random variable with distribution

$$\begin{aligned} g(z) &= h(x)\left|\frac{dx}{dz}\right|_{x=\sqrt{z}} + h(x)\left|\frac{dx}{dz}\right|_{x=-\sqrt{z}} \\ &= P(X|\{a<Y<b\})\left|\frac{dx}{dz}\right|_{x=\sqrt{z}} + P(X|\{a<Y<b\})\left|\frac{dx}{dz}\right|_{x=-\sqrt{z}} \\ &= \frac{1}{P(\{a<Y<b\})}\frac{\phi(\sqrt{z})}{2\sqrt{z}}\int_a^b \frac{\left(\exp\left[-\frac{1}{2(1-\rho^2)}(y-\rho\sqrt{z})\right]+\exp\left[-\frac{1}{2(1-\rho^2)}(y+\rho\sqrt{z})\right]\right)}{\sqrt{2\pi(1-\rho^2)}}dy \end{aligned}$$

$$SkQ[z] = \frac{\phi[\sqrt{z}]\left(\Phi\left[\frac{b-\rho\sqrt{z}}{\sqrt{1-\rho^2}}\right]-\Phi\left[\frac{a-\rho\sqrt{z}}{\sqrt{1-\rho^2}}\right]+\Phi\left[\frac{b+\rho\sqrt{z}}{\sqrt{1-\rho^2}}\right]-\Phi\left[\frac{a+\rho\sqrt{z}}{\sqrt{1-\rho^2}}\right]\right)}{2\sqrt{z}(\Phi[b]-\Phi[a])}. \tag{2.8}$$

Relations

If $X \sim SkQ(a,b,\rho)$ and $Z \sim \chi^2_1$, then

(i) $a = 0$ and $b \to \infty \Rightarrow X \to Z$ (convergence in distribution).
(ii) $b = 0$ and $a \to -\infty \Rightarrow X \to Z$ (convergence in distribution).
(iii) $\rho = 0 \Rightarrow X \to Z$ (convergence in distribution), $\forall a, b \in \Re$

2.2 Other generalizations

There are other two generalizations that we will discuss now. Using Eq.(2.5) with $a = k$ and $b \to \infty$, we get the extended skew-normal distribution which was studied by Birnbaum (see Pourahmadi, 2007).

$$Sk[y] = \phi(y)\Phi\left[\frac{\rho y - k}{\sqrt{1-\rho^2}}\right](1-\Phi(k))^{-1}. \tag{2.9}$$

The mean of this distribution is given by

$$E[Y] = \frac{\rho[\exp\{-k^2/2\}]}{\sqrt{2\pi}\Phi(-k)}. \tag{2.10}$$

Using the Eq.(2.6) we get

$$M_Y[t] = \frac{\exp\{t^2/2\}}{2\Phi(-k)}\left[1 - \text{erf}\left[\frac{k-\rho t}{\sqrt{2}}\right]\right]. \tag{2.11}$$

Or equivalently

$$M_Y[t] = \exp\{t^2/2\}\frac{\Phi(\rho t - k)}{\Phi(-k)}. \tag{2.12}$$

Using the Eq.(2.5) with $a \to -\infty$ and $b = k$, we get

$$Sk[y] = \phi(y)\Phi\left[\frac{k - \rho y}{\sqrt{1-\rho^2}}\right](\Phi(k))^{-1}. \tag{2.13}$$

The mean of this distribution is

$$E[Y] = -\frac{\rho[\exp\{-k^2/2\}]}{\sqrt{2\pi}\Phi(k)}. \tag{2.14}$$

Using the Eq.(2.6), we get

$$M_Y[t] = \frac{\exp\{t^2/2\}}{2\Phi(k)}\left[1 + \text{erf}\left[\frac{k-\rho t}{\sqrt{2}}\right]\right]. \tag{2.15}$$

Or equivalently

$$M_Y[t] = \exp\{t^2/2\}\frac{\Phi(k - \rho t)}{\Phi(k)}. \tag{2.16}$$

2.3 Applications

Two applications involving real data are detailed is this section.

2.3.1 Temperature of the ecological reserve of IBGE (Brazil)

We have used the data of the maximum temperature in the climatological station of the IBGE ecological reserve in 1985 (IBGE web site: *http://www.recor.org.br/Estacao/estacao.html*). We have plotted the Q-Q plot of the data in Figure 2, in which we can see that the normal distribution does not provide a good approximation because the left tail of the real distribution of the data is longer than the tail of the normal distribution.

We can see that the normal distribution does not provide the best fit because of the skewness of the data. Thus, estimating the parameters μ, σ and c in Eq.(2.3) for the data with $\hat{\mu} = 28.155517$, $\hat{\sigma} = 3.276348$ and $\hat{c} = -1.883828$, we get a better approximation than using the normal distribution. We can see from Figures 3 that the skew-normal has a better fit to the empirical distribution.

For the normal distribution the average deviation about the empirical distribution is 0.0257 and the maximum deviation is 0.0683. For the Skew normal distribution the average deviation about the empirical distribution is 0.01499 and the maximum deviation is 0.0561. We used the Kolmogorov-Smirnov goodness of fit test to check the adjustment at 1% and 5% of significance.

2.3.2 Evaporation Data of the ecological reserve of IBGE (Brazil)

We used the evaporation data (mm) in the climatological station of the IBGE ecological reserve in 1988 for this application (IBGE web site). This data is very skewed, so we fitted this data with the generalized skew normal distribution given by (2.5) along with one location parameter and one scale parameter.

To estimate the parameters, we first adjusted the skew normal distribution and then, using these initial parameters, we computationally maximized the likelihood function, to a few decimal places

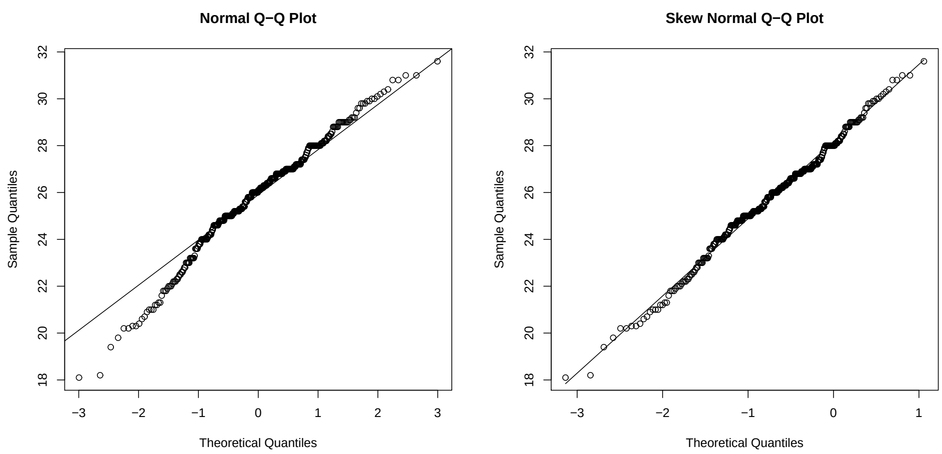

Figure 2: *Quantile-Quantile Plot of the normal distribution (left) and skew normal distribution (right)*

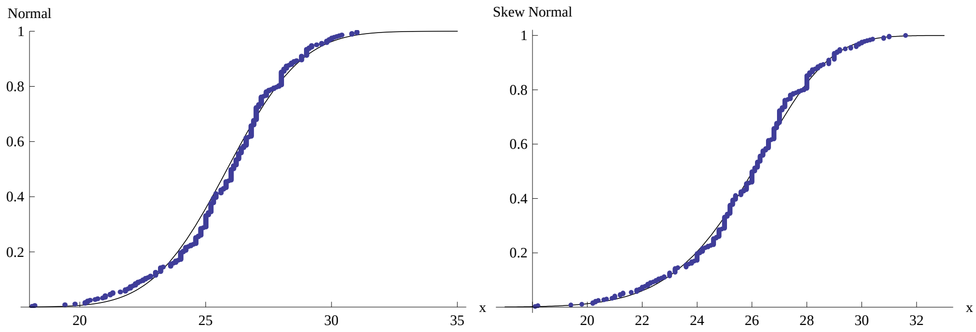

Figure 3: *Fit for the normal distribution (Left) and skew normal distribution (right).*

of precision. This gave the estimation of the parameters as $\hat{\mu} = 0.3$, $\hat{\sigma} = 4.1$, $\hat{a} = 0.2$, $\hat{b} = 2.8$, $\hat{\rho} = 0.99$. Figure 4 (left) shows the fit of the generalized skew normal distribution for the empirical distribution of the data, and Figure 4 (right) shows the fit of the normal distribution. The average deviation about the empirical distribution is $1.214\ 10^{-2}$, the maximum deviation is $4.087\ 10^{-2}$ and the mean square error is $2.21\ 10^{-4}$. We used the Kolmogorov-Smirnov goodness of fit test to check the adjustment at 1% and 5% of significance.

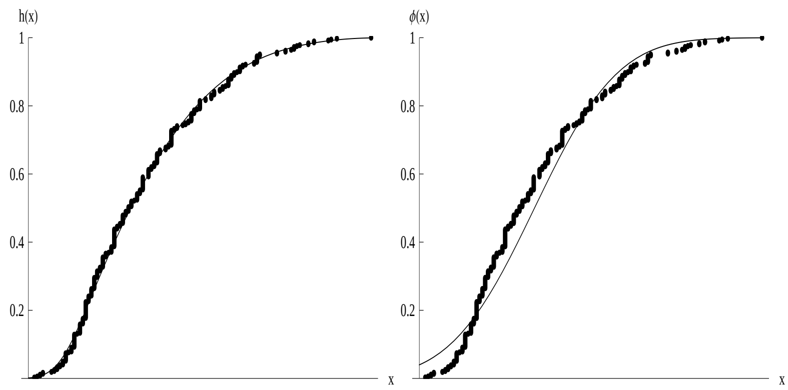

Figure 4: *Fit of the generalized skew normal distribution (left) and normal distribution (right)*

3 Generalization of Fernández and Steel skew distribution

Fernández and Steel (1998) introduced the following expression (See also Steel et al., 2006) for generating skew density functions

$$h(x) = \frac{2}{\alpha + 1/\alpha} f(x\alpha^{sign[x]}). \tag{3.1}$$

where $\alpha > 0, x \in (-\infty, \infty)$, and $f(x)$ is any symmetric density function.

The skew density in (3.1) is generalized here by taking

$$h(x) = cf(xa^{\frac{1+|x|/x}{2}} b^{\frac{1-|x|/x}{2}}). \tag{3.2}$$

where $a, b > 0, x \in (-\infty, \infty)$, and $f(x)$ is any symmetric density function. Since $h(x)$ is a density function, hence we get

$$c = \frac{2ab}{a+b}. \tag{3.3}$$

Clearly, for $a = b^{-1} = \alpha$, (3.2) reduces to (3.1). If we take $a = b$ in (3.2), we arrive at the following symmetric density function $h(x) = af(ax)$, where $f(x)$ is a symmetric density function of X. The n^{th} moment about the origin for (3.2) is given by

$$E(X^n) = \frac{2((-1)^n.a^{n+1} + b^{n+1})}{(a+b)(ab)^n} \int_0^\infty x^n f(x)dx. \tag{3.4}$$

The cumulative distribution function for (3.2) is

$$\begin{aligned} H(x) &= \begin{cases} 2aF(bx)/(a+b), x < 0 \\ 2ab\left[\frac{1}{2b} + \frac{1}{a}(F(ax) - 1/2)\right]/(a+b), x > 0 \end{cases} \\ &= \frac{2ab}{a+b}\left[\frac{1}{b}F\left(\frac{b}{2}(x-|x|)\right) + \frac{1}{a}F\left(\frac{a}{2}(x+|x|)\right) - \frac{1}{2a}\right], x \in (-\infty, \infty). \end{aligned} \tag{3.5}$$

where $F(x)$ is the distribution function corresponding to $f(x)$.

3.1 Renyi and Shannon entropies and Fisher information

The Shannon entropy (Thomas et al., 2006) for (3.2) is

$$Sh(X) = -\int_{-\infty}^{\infty} h(x). \ln h(x)dx = Sh_f(X) - \ln(c), \tag{3.6}$$

where $Sh_f(X)$ is the Shannon entropy for the symmetric density function $f(x)$, and c is as given in (3.3). The Renyi entropy (Thomas et al., 2006) for (3.2) is

$$R_\alpha(X) = \frac{1}{1-\alpha} \ln\left[\int_{-\infty}^{\infty} [h(x)]^\alpha dx\right] = R_f(X) - \ln(c), \tag{3.7}$$

where $R_f(X)$ is the Renyi entropy for the symmetric density function $f(x)$.

The Fisher information (Thomas et al., 2006) for (3.2) which depends on a parameter θ of $f(x)$

is

$$
\begin{aligned}
I(\theta) &= E\left[\left(\frac{d}{d\theta}\ln h(x)\right)^2\right] \\
&= \int_{\infty}^{\infty} c\left(\frac{d}{d\theta}\ln f(xa^{\frac{1+|x|/x}{2}}b^{\frac{1-|x|/x}{2}})\right)^2 f(xa^{\frac{1+|x|/x}{2}}b^{\frac{1-|x|/x}{2}})dx \\
&= c\int_{-\infty}^{0}\left(\frac{d}{d\theta}\ln f(xb)\right)^2 f(xb)dx + c\int_{0}^{\infty}\left(\frac{d}{d\theta}\ln f(xa)\right)^2 f(xa)dx = IF(\theta), \qquad (3.8)
\end{aligned}
$$

where $IF(\theta)$ is the Fisher information for the parameter θ of the symmetric density function $f(x)$.

The Fisher information for the parameter a of the distribution $h(x)$ given by (3.2) is

$$
I(a) = k^2 + 2kc\int_0^{\infty}\frac{d}{da}f(ax)dx + c\int_0^{\infty}\left[\frac{d}{da}\ln f(ax)\right]^2 f(ax)dx, \qquad (3.9)
$$

where $k = b/(a(a+b))$ and c is given by (3.3). The Fisher information for the parameter b of the distribution $h(x)$ given by (3.2) is

$$
I(b) = k^2 + 2kc\int_{-\infty}^{0}\frac{d}{db}f(bx)dx + c\int_{-\infty}^{0}\left[\frac{d}{db}\ln f(bx)\right]^2 f(bx)dx, \qquad (3.10)
$$

where $k = b/(a(a+b))$ and c is given by (3.3).

The Fisher information for X, of the distribution $h(x)$ given by (3.2), is

$$
I(X) = abIF(X), \qquad (3.11)
$$

where $IF(X)$ is the Fisher information for the symmetric density function $f(x)$.

3.2 Asymmetric normal distribution

Taking the normal distribution in (3.2), we have the following asymmetric normal distribution

$$
h(x) = \frac{2ab}{\sqrt{2\pi}(a+b)}\exp\left[-\frac{x^2}{2}a^{1+|x|/x}b^{1-|x|/x}\right]. \qquad (3.12)
$$

A few graphs for the asymmetric normal density function (3.12) and its distribution function obtained using (3.5), for certain values of a and b, are given in Figures 5 and 6. For a=b, (3.12) reduces to the normal symmetric density function

$$
h(x) = a\phi(ax) = ae^{-x^2a^2/2}/\sqrt{2\pi}, a > 0, x \in (-\infty, \infty). \qquad (3.13)
$$

The n^{th} moments for (3.12), utilizing (3.4), is given by

$$
E(X^n) = \frac{2^{n/2}((-1)^n.a^{n+1} + b^{n+1})}{(a+b)(ab)^n}\frac{\Gamma((n+1)/2)}{\sqrt{\pi}}. \qquad (3.14)
$$

The Moment Generating function for (3.12) is

$$M_X(t) = \frac{a}{a+b}e^{t^2/2b^2}\left[1 - \text{erf}\left[\frac{t}{\sqrt{2}b}\right]\right] + \frac{b}{a+b}e^{t^2/2a^2}\left[1 + \text{erf}\left[\frac{t}{\sqrt{2}a}\right]\right] \tag{3.15}$$

or equivalently

$$M_X(t) = \frac{2a}{a+b}e^{t^2/2b^2}\Phi(-t/b) + \frac{2b}{a+b}e^{t^2/2a^2}\Phi(t/a). \tag{3.16}$$

The mean and variance are given by

$$E(X) = \frac{2(b-a)}{ab}\int_0^\infty \frac{xe^{-x^2/2}}{\sqrt{2\pi}}dx = \frac{2(b-a)}{\sqrt{2\pi}ab} \tag{3.17}$$

$$Var(X) = \frac{1}{a^2b^2}\left[\frac{a^3+b^3}{a+b} - \frac{2(b-a)^2}{\pi}\right]. \tag{3.18}$$

Consider that we have a sample $X_1, X_2, \dots, X_n$ and we want to estimate a and b. Let $X_1, X_2, \dots, X_{n'}$ be the negative observations and $X_{n'+1}, \dots, X_n$ the positive observations. Thus, the maximum likelihood estimator (Bickel et al., 2007) $\hat{a}$ for a is the solution of the equation

$$\hat{a}^3 \sum_{i=n'+1}^{n} X_i^2 + \hat{a}^2 b \sum_{i=n'+1}^{n} X_i^2 = nb. \tag{3.19}$$

The maximum likelihood estimator $\hat{b}$ for b is the solution of the equation

$$\hat{b}^3 \sum_{i=0}^{n'} X_i^2 + \hat{b}^2 a \sum_{i=0}^{n'} X_i^2 = na. \tag{3.20}$$

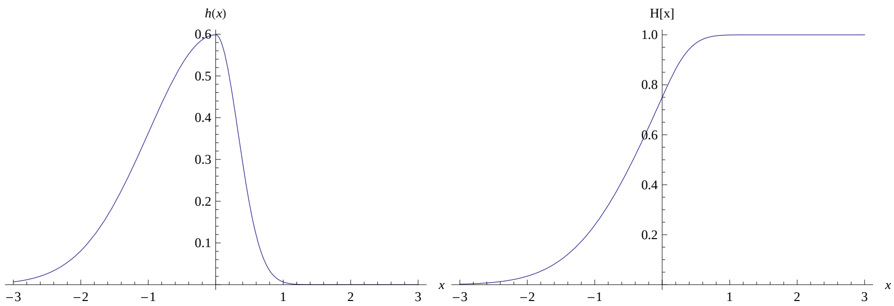

Figure 5: *Asymmetric normal (PDF and CDF)* $a = 3$ *and* $b = 1$

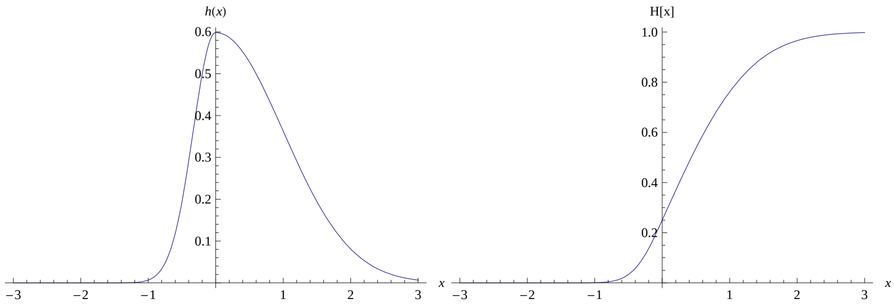

Figure 6: *Asymmetric normal (PDF and CDF)* $a = 1$ *and* $b = 3$

3.2.1 Renyi and Shannon entropies and Fisher information for the asymmetric normal distribution

The Shannon Entropy for the distribution given by (3.12), using (3.6), is

$$Sh(X) = \frac{1}{2}\ln(2\pi e) - \ln(c). \tag{3.21}$$

The Renyi Entropy for this distribution, using (3.7) is

$$R_\alpha(X) = \frac{1}{2}\ln(2\pi) - \frac{\ln(\alpha)}{2(1-\alpha)} - \ln(c). \tag{3.22}$$

The Fisher information for a, b and X, using (3.12), (3.9), (3.10) and (3.11) respectively, are

$$I(a) = \frac{b(3a+2b)}{a^2(a+b)^2}, \quad I(b) = \frac{a(3b+2a)}{b^2(b+a)^2}, \quad I(X) = ab. \tag{3.23}$$

We calculated the relative Fisher information (Yáńez, 2008) which is defined for two probability densities $\rho_1(X)$ and $\rho_2(X)$ by

$$I(p_1,p_2) = \int_R f_1(x)\left[\frac{d}{dx}\ln\left(\frac{f_1(x)}{f_2(x)}\right)\right]^2 dx. \tag{3.24}$$

Let $\rho_1(X)$ be the distribution given by (3.12) with parameters a and b and $\rho_2(X)$ be the distribution given by (3.12) with parameters c and d, thus we have the relative Fisher information as

$$I(\rho_1,\rho_2) = \frac{a^3\left(b^2-d^2\right)^2 + b^3\left(a^2-c^2\right)^2}{a^2b^2(a+b)}. \tag{3.25}$$

We have plotted the Relative Fisher information function for the parameters a, b, c, d in Figure 7 and Figure 8 .

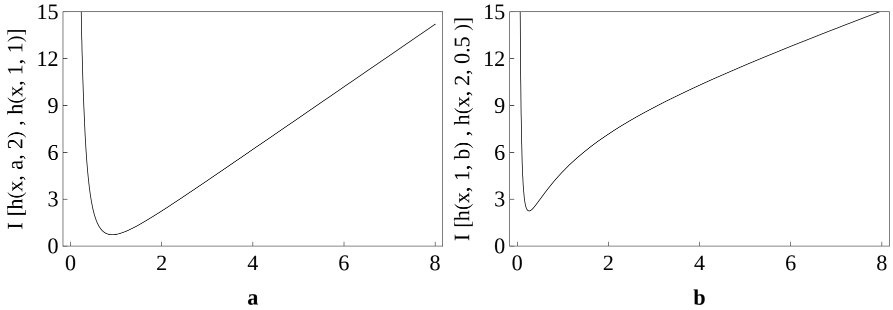

Figure 7: $I(\rho_1, \rho_2)$ *for* $b = 2$, $c = 1$, $d = 1$, *and* $I(\rho_1, \rho_2)$ *for* $a = 1$, $c = 2$, $d = 0.5$

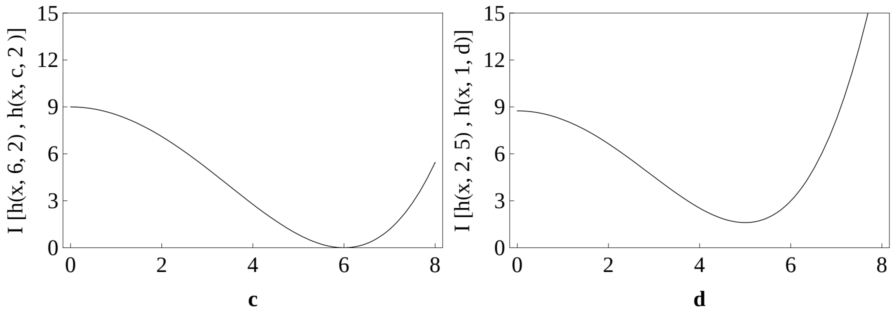

Figure 8: $I(\rho_1, \rho_2)$ *for* $a = 6$, $b = 2$, $d = 2$, *and* $I(\rho_1, \rho_2)$ *for* $a = 2$, $b = 5$, $c = 1$

3.2.2 Distribution of X^2 (another generalization for the chi-square distribution)

If the random variable X has distribution $h(x)$, then $Y = X^2$ has probability density function, mean, moments, cumulative distribution function and moment generating function given, respectively, by

- PDF:

$$aq(y) = \frac{ab}{(a+b)\sqrt{y}}[f(a\sqrt{y}) + f(b\sqrt{y})], \qquad y \in (0, \infty). \tag{3.26}$$

- Mean:

$$\frac{a^3 + b^3}{a^2 b^2 (a+b)}. \tag{3.27}$$

- Moments:

$$E(X^n) = \frac{2^n \Gamma[n+1/2]}{(a+b)\sqrt{\pi}} \left(\frac{a}{b^{2n}} + \frac{b}{a^{2n}} \right). \tag{3.28}$$

- CDF:

$$AQ(y) = \frac{1}{(a+b)} \left\{ b.\ \text{erf}\left[\frac{a\sqrt{y}}{\sqrt{2}}\right] + a.\ \text{erf}\left[\frac{b\sqrt{y}}{\sqrt{2}}\right] \right\}, \qquad y \in (0, \infty). \tag{3.29}$$

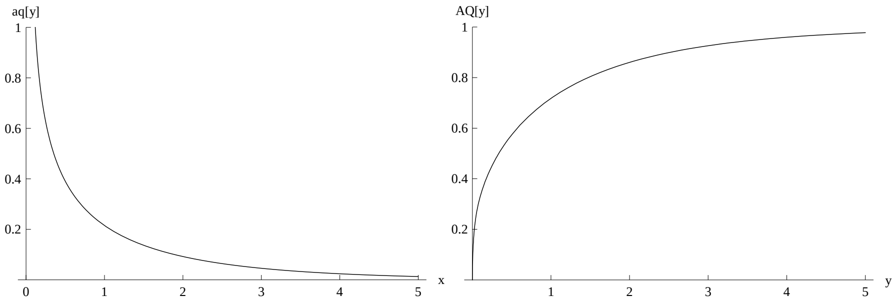

Figure 9: $aq(y)$ *and* $AQ(y)$ *for* $a = 8$ *and* $b = 1$

- Moment Generating Function:

$$M_Y(t) = \frac{ab}{(a+b)} \left\{ \frac{1}{\sqrt{a^2 - 2t}} + \frac{1}{\sqrt{b^2 - 2t}} \right\}, \qquad t < a^2/2 \text{ and } t < b^2/2. \tag{3.30}$$

We plotted the density function given by (3.26) and the distribution function given by (3.29) in Figure 9.

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