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ON RELATIONS FOR GENERALIZED RAYLEIGH DISTRIBUTION BASED ON LOWER GENERALIZED ORDER STATISTICS AND ITS CHARACTERIZATION

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SUMMARY

In this paper, we derive the explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics from generalized Rayleigh distribution. The results include as particular cases the above relations for moments of order statistics and lower records. Further, using a recurrence relation for single moments we obtain characterization of generalized Rayleigh distribution.

Keywords and phrases: Lower generalized order statistics, order statistics, lower records, single moments, product moments, recurrence relations, generalized Rayleigh distribution and characterization.

AMS Classification: 62G30, 62E10

1 Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995) as a unified approach to different model e.g. usual order statistics, sequential order statistics, Stigler's order statistics, record values. They can be easily applicable in practice problems except when $F()$ is so called inverse distribution function. For this, when $F()$ is an inverse distribution function, we need a concept of lower generalized order statistics ($lgos$), which is given as:

Let $n \in N$, $k \geq 1$, $m \in \Re$, be such that $\gamma_r = k + (n-r)(m+1) > 0$, for all $1 \leq r \leq n$. By the $lgos$ from an absolutely continuous distribution function (df) $F(x)$ with the probability density function pdf $f(x)$ we mean random variables $X'(1,n,m,k)$,...,$X'(n,n,m,k)$ having joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq \cdots \geq x_n > F^{-1}(0)$. For simplicity we shall assume $m_i = m$, $\forall i = 1,\ldots,n-1$. The pdf of the r^{th} $lgos$, is given by

$$f_{X'(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1}f(x)g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint pdf of r^{th} and s^{th} $lgos$, $1 \le r < s \le n$ is

$$f_{X'(r,n,m,k),X'(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}(k + (n-i)(m+1)), \quad h_m(x) = \begin{cases} -\ln x, & m = -1 \\ -\frac{x^{m+1}}{m+1}, & m \neq -1 \end{cases}$$
$$\text{and} \quad g_m(x) = h_m(x) - h_m(1), \ \ x \in [0,1).$$

We shall also take $X'(0,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X'(r,n,m,k)$ reduced to the $(n-r+1)-$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X'(r,n,m,k)$ reduced to the $r-$ th lower k record value (Pawlas and Szynal, 2001). The work of Burkschat et al. (2003) may also refer for lower generalized order statistics.

Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution are derived by Pawlas and Szynal (2001). Khan $et\ al.$ (2008) have established recurrence relations for moments of lower generalized order statistics from exponentiated Weibull distribution, Khan and Kumar (2010) have established recurrence relations for moments of lower generalized order statistics from exponentiated Pareto distribution. Ahsanullah (2004) and Mbah and Ahsanullah (2007) characterized the uniform and power function distributions based on distributional properties of lower generalized order statistics, respectively.

Characterizations of particular distributions based on the moments and conditional moments of order statistics were presented by some authors such as Wu and Ouyang (1996), Grudzien and Szynal (1998), Khan and Abouammoh (1999), Ahmad (2001), Asadi $et\ al.$ (2001), Govindarajulu (2001), among others. Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization.

In this paper, we have established explicit expressions and some recurrence relations for moments of $lgos$ from generalized Rayleigh distribution. Result for order statistics and lower record values are deduced as special cases and a characterization of generalized Rayleigh distribution has been obtained on using a recurrence relation for single moments.

A random variable X is said to have generalized Rayleigh distribution (Surles and Padgett, 2001) if its pdf is of the form

$$f(x) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}(1 - e^{-(\lambda x)^2})^{\alpha-1}, \ \ x > 0, \ \alpha, \lambda > 0 \tag{1.4}$$

and the corresponding df is

$$F(x) = (1 - e^{-(\lambda x)^2})^{\alpha}, \ \ x > 0, \ \alpha, \lambda > 0. \tag{1.5}$$

2 Relations for single moments

Note that for generalized Rayleigh distribution

$$F(x) = \frac{(e^{(\lambda x)^2} - 1)}{2\alpha\lambda^2 x} f(x). \tag{2.1}$$

The *pdf* (1.2) can be written for the generalized Rayleigh distribution with *pdf* (1.4) and *df* (1.5) in the following form

$$f_{X'(r,n,m,k)}(x) = \begin{cases} \frac{2\alpha^r k^r \lambda^2 x e^{-(\lambda x)^2}}{(r-1)!}(1-e^{-(\lambda x)^2})^{\alpha k-1}[-\ln(1-e^{-(\lambda x)^2})]^{r-1}, & m=-1 \\ \frac{2\alpha\lambda^2 C_{r-1}}{(r-1)!(m+1)^{r-1}} x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\alpha\gamma_r-1} & \\ \times[1-(1-e^{-(\lambda x)^2})^{\alpha(m+1)}]^{r-1}, & m \neq -1 \end{cases} \tag{2.2}$$

By using binomial and logarithmic expansions, we can rewrite (2.2) as

$$f_{X'(r,n,m,k)}(x) = \begin{cases} \frac{2\alpha^r k^r \lambda^2}{(r-1)!} \sum_{t=0}^{\infty}\sum_{a=0}^{\infty}(-1)^a \phi_t(r-1)\binom{\alpha k-1}{a} x e^{-(a+r+t)(\lambda x)^2}, & m=-1 \\ \Psi(a)\binom{r-1}{a}\binom{\alpha(\gamma_r+(m+1)a)-1}{b} x e^{-(b+1)(\lambda x)^2}, & m\neq -1 \end{cases} \tag{2.3}$$

where

$$\Psi(a) = \frac{2\alpha\lambda^2 C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1}\sum_{b=0}^{\infty}(-1)^{a+b}$$

and $\phi_t(r-1)$ is the coefficient of $e^{-(r-1+t)(\lambda x)^2}$ in the expansion of $\left(\sum_{t=1}^{\infty} e^{-t(\lambda x)^2}(1/t)\right)^{r-1}$ (see Balakrishnan and Cohan, 1991; Shawky and Bakoban, 2008). We shall first establish the explicit formula for $E[X'(r,n,m,k)]$. Using (2.3), we obtain when $m \neq -1$

$$\begin{aligned} E[X'^j(r,n,m,k)] &= \frac{2\alpha\lambda^2 C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{a=0}^{r-1}\sum_{b=0}^{\infty}(-1)^{a+b}\binom{r-1}{a} \\ &\quad \times \binom{\alpha(\gamma_r+(m+1)a)-1}{b} \int_0^{\infty} x^{j+1} e^{-(b+1)(\lambda x)^2} dx \\ &= \frac{\alpha C_{r-1}}{\lambda^j (r-1)! \lambda^j (m+1)^{r-1}} \sum_{a=0}^{r-1}\sum_{b=0}^{\infty}(-1)^{a+b}\binom{r-1}{a} \\ &\quad \times \binom{\alpha(\gamma_r+(m+1)a)-1}{b} \frac{\Gamma((j/2)+1)}{(b+1)^{(j/2)+1}} \end{aligned} \tag{2.4}$$

and when $m = -1$ that

$$\begin{aligned} E[X'^j(r,n,-1,k)] &= \frac{2(\alpha k)^r \lambda^2}{(r-1)!} \sum_{t=0}^{\infty}\sum_{a=0}^{\infty}(-1)^a \phi_t(r-1)\binom{\alpha k-1}{a} \int_0^{\infty} x^{j+1} e^{-(a+r+t)(\lambda x)^2} dx \\ &= \frac{(\alpha k)^r}{\lambda^j (r-1)!} \sum_{t=0}^{\infty}\sum_{a=0}^{\infty}(-1)^a \phi_t(r-1)\binom{\alpha k-1}{a} \frac{\Gamma((j/2)+1)}{(a+r+t)^{(j/2)+1}}. \end{aligned} \tag{2.5}$$

If α is a positive integer, the relations (2.4) and (2.5) then give

$$E[X'^j(r,n,m,k)] = \frac{\alpha C_{r-1}}{\lambda^j (r-1)! \lambda^j (m+1)^{r-1}} \sum_{a=0}^{r-1} \sum_{b=0}^{\alpha(\gamma_r+(m+1)a)-1} (-1)^{a+b} \binom{r-1}{a} \times \binom{\alpha(\gamma_r+(m+1)a)-1}{b} \frac{\Gamma((j/2)+1)}{(b+1)^{(j/2)+1}} \tag{2.6}$$

and

$$E[X'^j(r,n,-1,k)] = \frac{(\alpha k)^r}{\lambda^j (r-1)!} \sum_{t=0}^{\infty} \sum_{a=0}^{\alpha k-1} (-1)^a \phi_t(r-1) \binom{\alpha k-1}{a} \frac{\Gamma((j/2)+1)}{(a+r+t)^{(j/2)+1}}. \tag{2.7}$$

Special cases

(i) Putting $m=0,\ k=1$ in (2.6), the explicit formula for single moments of order statistics of the generalized Rayleigh distribution can be obtained as

$$E(X^j_{n-r+1:n}) = \frac{\alpha C_{r:n}}{\lambda^j} \sum_{a=0}^{r-1} \sum_{b=0}^{\alpha(n-r+1+a)-1} (-1)^{a+b} \binom{r-1}{a} \binom{\alpha(n-r+1+a)-1}{b} \frac{\Gamma((j/2)+1)}{(b+1)^{(j/2)+1}},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

(ii) Putting $k=1$ in (2.7), we deduce the explicit expression for the moments of lower record values for the generalized exponential distribution as

$$E[X'^j(r,n,-1,1)] = \frac{\alpha^r}{\lambda^j (r-1)!} \sum_{t=0}^{\infty} \sum_{a=0}^{\alpha-1} (-1)^a \phi_t(r-1) \binom{\alpha-1}{a} \frac{\Gamma((j/2)+1)}{(a+r+t)^{(j/2)+1}}.$$

A recurrence relation for single moments of $lgos$ from df (1.5) can be obtained in the following theorem.

Theorem 1. *For the distribution given in (1.5) and for* $2 \le r \le n$, $n \ge 2$ *and* $k = 1, 2, \ldots$

$$E[X'^j(r,n,m,k)] - E[X'^j(r-1,n,m,k)] = \frac{j}{2\alpha\lambda^2\gamma_r} \{E[X'^{j-2}(r,n,m,k)] - E[\phi(X'(r,n,m,k))]\}, \tag{2.8}$$

where $\phi(x) = x^{j-2} e^{(\lambda x)^2}$.

Proof. From (1.2), we have

$$E[X'^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{2.9}$$

Integrating by parts treating $[F(x)]^{\gamma_r - 1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$E[X'^j(r,n,m,k)] = E[X'^j(r-1,n,m,k)] - \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx,$$

the constant of integration vanishes since the integral considered in (2.9) is a definite integral. On using (2.1), we obtain

$$\begin{aligned} E[X'^j(r,n,m,k)] = E[X'^j(r-1,n,m,k)] - \frac{jC_{r-1}}{2\alpha\lambda^2\gamma_r(r-1)!} \\ \times \{\int_0^\infty x^{j-2} e^{(\lambda x)^2}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx \\ - \int_0^\infty x^{j-2}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx\} \end{aligned} \tag{2.10}$$

and hence the result. □

Remark 1. Putting $m = 0$, $k = 1$, in (2.8), we obtain a recurrence relation for single moments of order statistics of the generalized Rayleigh distribution in the form

$$E(X^j_{n-r+1:n}) = E(X^j_{n-r+2:n}) + \frac{j}{2\alpha\lambda^2(n-r+1)}\{E(X^{j-2}_{n-r+1:n}) - E(\phi(X_{n-r+1:n}))\}.$$

Remark 2. Setting $m = -1$ and $k \geq 1$, in Theorem 1, we get a recurrence relation for single moments of lower $k-$ th record values from generalized Rayleigh distribution in the form

$$E[X'^j(r,n,-1,k)] = E[X'^j(r-1,n,-1,k)] + \frac{j}{2\alpha\lambda^2 k}\{E[X'^{j-2}(r,n,-1,k)] - E[\phi(X'(r,n,-1,k))]\}.$$

3 Relations for product moments

Making use of (2.1), we can derive recurrence relations for product moments of $lgos$ from (1.5).

Theorem 2. *For the distribution given in (1.5) and for* $1 \leq r < s \leq n$, $n \geq 2$ *and* $k = 1, 2, \ldots$

$$\begin{aligned} E[X'^i(r,n,m,k)X'^j(s,n,m,k)] &= E[X'^i(r,n,m,k)X'^j(s-1,n,m,k)] \\ &+ \frac{j}{2\alpha\lambda^2\gamma_s}\{E[X'^i(r,n,m,k)X'^{j-2}(s,n,m,k)] \\ &- E[\phi(X'(r,n,m,k)X'(s,n,m,k))]\}, \end{aligned} \tag{3.1}$$

where $\phi(x,y) = x^i y^{j-2} e^{(\lambda y)^2}$.

Proof. From (1.3), we have

$$E[X'^i(r,n,m,k)X'^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i [F(x)]^m f(x) g_m^{r-1}(F(x)) I(x)dx, \tag{3.2}$$

where

$$I(x)=\int_0^x y^j[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.2), we get

$$\begin{aligned}E[X'^i(r,n,m,k)X'^j(s,n,m,k)] &= E[X'^i(r,n,m,k)X'^j(s-1,n,m,k)]\\ &\quad-\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^x x^iy^{j-1}[F(x)]^mf(x)g_m^{r-1}(F(x))\\ &\quad\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}dydx\end{aligned}$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (2.1), we obtain

$$\begin{aligned}&E[X'^i(r,n,m,k)X'^j(s,n,m,k)]\\ &=E[X'^i(r,n,m,k)X'^j(s-1,n,m,k)-\frac{jC_{s-1}}{2\alpha\lambda^2\gamma_s(r-1)!(s-r-1)!}\\ &\quad\{\int_0^\infty\int_0^x x^iy^{j-2}e^{(\lambda y)^2}[F(x)]^mf(x)g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx\\ &\quad-\int_0^\infty\int_0^x x^iy^{j-2}[F(x)]^mf(x)g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx\}\end{aligned}$$

and hence the result. □

Remark 3. Putting $m=0,\ k=1$ in (3.1), we obtain recurrence relations for product moments of order statistics of the generalized Rayleigh distribution in the form

$$\begin{aligned}E(X^i_{n-r+1:n}X^j_{n-s+1:n}) &= E(X^i_{n-r+1:n}X^j_{n-s+2:n})+\frac{j}{2\alpha\lambda^2(n-s+1)}\\ &\quad\times\ \left\{E(X^i_{n-r+1:n}X^{j-2}_{n-s+1:n})-E(\phi(X_{n-r+1:n}X_{n-s+1:n})\right\}.\end{aligned}$$

Remark 4. Setting $m=-1$ and $k\geq 1$ in Theorem 2, we obtain the recurrence relations for product moments of lower $k-$ th record values from generalized Rayleigh distribution in the form

$$\begin{aligned}E[X'^i(r,n,-1,k)X'^j(s,n,-1,k)] &= E[X'^i(r,n,-1,k)X'^j(s-1,n,-1,k)]\\ &\quad+\frac{j}{2\alpha\lambda^2k}\{E[X'^i(r,n,-1,k)X'^{j-2}(s,n,-1,k)]\\ &\quad-E[\phi(X'(r,n,-1,k)X'(s,n,-1,k))]\}.\end{aligned}$$

4 Characterization

Theorem 3. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$E[X'^j(r,n,m,k)] = E[X'^j(r-1,n,m,k)] - \frac{j}{2\alpha\lambda^2\gamma_r}E[\phi(X'(r,n,m,k))] + \frac{j}{2\alpha\lambda^2\gamma_r}E[X'^{j-2}(r,n,m,k)] \tag{4.1}$$

if and only if

$$F(x) = (1 - e^{-(\lambda x)^2})^\alpha, \ \ x > 0, \ \alpha, \lambda > 0.$$

Proof. The necessary part follows immediately from equation (2.8). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.2), we have

$$\begin{aligned} \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\ = \frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx \\ - \frac{jC_{r-1}}{2\alpha\lambda^2\gamma_r(r-1)!}\int_0^\infty x^{j-2}e^{(\lambda x)^2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\ + \frac{jC_{r-1}}{2\alpha\lambda^2\gamma_r(r-1)!}\int_0^\infty x^{j-2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \end{aligned} \tag{4.2}$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\begin{aligned} \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\ = \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx \\ + \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\ - \frac{jC_{r-1}}{2\alpha\lambda^2\gamma_r(r-1)!}\int_0^\infty x^{j-2}e^{(\lambda x)^2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\ + \frac{jC_{r-1}}{2\alpha\lambda^2\gamma_r(r-1)!}\int_0^\infty x^{j-2}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \end{aligned}$$

which reduces to

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\{F(x) + \frac{1}{2\alpha\lambda^2 x}f(x) - \frac{e^{(\lambda x)^2}}{2\alpha\lambda^2 x}f(x)\}dx = 0. \tag{4.3}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (4.3), we get

$$\frac{f(x)}{F(x)} = \frac{2\alpha\lambda^2 x}{(e^{(\lambda x)^2} - 1)}$$

which prove that $F(x) = (1 - e^{-(\lambda x)^2})^\alpha, \ x > 0, \ \alpha, \lambda > 0.$ $\square$

5 Applications

The results established in this paper can be used to determine the mean, variance and coefficients of skewness and kurtosis. The moments can also be used for finding best linear unbiased estimator (BLUE) for parameter and conditional moments. Some of the results are then used to characterize the distribution.

6 Conclusion

This paper deals with the lower generalized order statistics from the generalized Rayleigh distribution. Explicit expressions and recurrence relations between the single and product moments are derived. Characterization of the generalized Rayleigh distribution based on a recurrence relation for single moments is discussed. Special cases are also deduced.

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References

[1] Ahmad, A.A. (2001). Moments of order statistics from doubly truncated continuous distributions and characterization. *Statistics*, **35**, 479-494.

[2] Ahsanullah, M. (2004). A characterization of the uniform distribution by dual generalized order statistics. *Comm. Statist. Theory Methods*, **33**, 2921-2928.

[3] Asadi M., Rao, C.R. and Shanbahag, D.N. (2001). Some unified characterization result on generalized Pareto distribution. *J. Statist. Plann. Inference*, **93**, 29-50.

[4] Balakrishnan, N. and Cohan, A.C. (1991). *Order Statistics and Inference: Estimation Methods.* Academic Press, San Diego.

[5] Burkschat, M., Cramer, E. and Kamps, U. (2003). Dual generalized order statistics. *Metron,* **LXI**, 13-26.

[6] Govindarajulu, Z. (2001). Characterization of double exponential using moments of order statistics. *Comm. Statist. Theory Methods,* **30**, 2355-2372.

[7] Grudzien, Z. and Szynal, D. (1998). On characterizations of continuous in terms of moments of order statistics when the sample size is random. *J. Math. Sci.,* **92**, 4017-4022.

[8] Hwang, J.S. and Lin, G.D. (1984). On a generalized moments problem II. *Proc. Amer. Math. Soc.*, **91**, 577-580.

[9] Kamps, U. (1995). *A Concept of Generalized Order Statistics.* B.G. Teubner Stuttgart.

[10] Kamps, U. (1998). Characterizations of distributions by recurrence relations and identities for moments of order statistics. In: Balakrishnan, N. and Rao, C.R., *Handbook of Statistics* **16**, *Order Statistics: Theory and Methods,* North-Holland, Amsterdam, 291-311.

[11] Khan, A.H. and Abouammoh, A.M. (1999). Characterizations of distributions by conditional expectation of order statistics. *J. Appl. Statist. Sci.,* **9**, 159-168.

[12] Khan, R.U., Anwar, Z. and Athar, H. (2008). Recurrence relations for single and product moments of dual generalized order statistics from exponentiated Weibull distribution.*Aligarh J. Statist.,* **28**, 37-45.

[13] Khan, R.U. and Kumar, D. (2010). On moments of lower generalized order statistics from exponentiated Pareto distribution and its characterization. *Appl. Math. Sci.* (Ruse), **4**, 2711-2722.

[14] Mbah, A.K. and Ahsanullah, M. (2007). Some characterization of the power function distribution based on lower generalized order statistics. *Pakistan J. Statist.,* **23**, 139-146.

[15] Pawlas, P. and Syznal, D. (2001). Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. *Demonstration Math.*, **XXXIV**, 353-358.

[16] Shawky, A.I. and Bakoban R.A. (2008). Characterization from exponentiated gamma distribution based on record values. *J. Stat. Theory Appl.,* **7**, 263-277.

[17] Surles, J.G. and Padgtt, W.J. (2001). Inference for reliability and stress-strength for a scaled Burr type X distribution. *Lifetime data Anal.,* **7**, 187-200.

[18] Wu, J. and Ouyang, L.Y., (1999). On characterizing distributions by conditional expectations of functions of order statistics. *Metrika,* **34**, 135-147.