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SEVERAL SIMPLE LINEAR MODELS WITH MEASUREMENT ERRORS: PARALLELISM HYPOTHESIS

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SUMMARY

The problem of estimation of the regression parameters in several simple regression models with measurement errors is considered when it is suspected that the regression lines may be parallel with some degree of uncertainty. In this regard we propose five estimators, namely, (i) the unrestricted estimator, (ii) the restricted estimator, (iii) the preliminary test estimator, (iv) the Stein-type estimator and (v) the positive-rule Stein-type estimator as in Saleh (2006). Properties of these estimators are studied in an asymptotic set up and the asymptotic distributional bias, MSE matrices, and risk under a quadratic loss function are obtained based on a sequence of local alternatives and dominance properties of these estimators are provided.

*Keywords and phrases:*local alternatives, measurement errors, Preliminary-test estimator, Stein-type estimators

1 Introduction

For several simple linear models with measurement errors, estimation of parameters is considered when it is suspected with some degree of uncertainty that the lines may be parallel. We consider p independent bivariate samples $\{(x_{\alpha_j}, Y^0_{\alpha_j})|\alpha = 1, \ldots, p, j = 1, \ldots, n_\alpha\}$ such that $Y^0_{\alpha_j} \sim \mathcal{N}(\theta_\alpha + \beta_\alpha \mu_{x_\alpha}, \sigma_{ee})$ for each pair (α_j, j), where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the slope and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$ is the intercept parameters. It is common to test the parallelism hypothesis $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ (where β_0 is an unknown scalar) against the alternative H_A : at least one pair of the components of $\boldsymbol{\beta}$ differ. Our problem is the estimation of the slope parameters when H_0 is suspected to hold. Consider simple linear models

$$\begin{cases} Y^0_{\alpha_j} = \theta_\alpha + \beta_\alpha x_{\alpha_j} + e_{\alpha_j} & j = 1, \ldots, n_\alpha \\ X^0_{\alpha_j} = x_{\alpha_j} + u_{\alpha_j}, & \alpha = 1, \ldots, p \end{cases} \tag{1.1}$$

where $e_{\alpha_j} \sim \mathcal{N}(0, \sigma_{ee})$ is the measurement error in the study variable and u_{α_j} is the measurement error in the explanatory variable. Note that x_{α_j} is unobservable and $X^0_{\alpha_j}$ is the corresponding observed value. Furthermore, we assume that

$$(x_{\alpha_j}, e_{\alpha_j}, u_{\alpha_j})' \sim \mathcal{N}_3 \left\{(\mu_{x_\alpha}, 0, 0) : \text{Diag}(\sigma_{x_\alpha x_\alpha}, \sigma_{ee}, \sigma_{u_\alpha u_\alpha})\right\}. \tag{1.2}$$

Then, $(Y^0_{\alpha_j}, X^0_{\alpha_j})'$ follows a bivariate normal distribution, that is,

$$\begin{pmatrix} Y^0_{\alpha_j} \\ X^0_{\alpha_j} \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_\alpha \end{pmatrix} : \mathbf{\Sigma}_{\alpha\alpha} \right\}, \tag{1.3}$$

where

$$\mathbf{\Sigma}_{\alpha\alpha} = \begin{pmatrix} \sigma_{Y^0_\alpha Y^0_\alpha} & \sigma_{X^0_\alpha Y^0_\alpha} \\ \sigma_{X^0_\alpha Y^0_\alpha} & \sigma_{X^0_\alpha X^0_\alpha} \end{pmatrix} = \begin{pmatrix} \beta^2_\alpha \sigma_{x_\alpha x_\alpha} + \sigma_{ee} & \beta_\alpha \sigma_{x_\alpha x_\alpha} \\ \beta_\alpha \sigma_{x_\alpha x_\alpha} & \sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha} \end{pmatrix}. \tag{1.4}$$

The conditional distribution of Y_{α_j} given X_{α_j} is again normal with mean and variance

$$E(Y^0_{\alpha_j} | X^0_{\alpha_j}) = \nu_{\alpha_0} + \nu_{\alpha_1} X_{\alpha_j}; \quad \sigma^{(\alpha)}_{zz} = \sigma_{ee} + \beta^2_\alpha \sigma_{x_\alpha x_\alpha}(1 - \kappa_{x_\alpha x_\alpha}), \tag{1.5}$$

where $\nu_{\alpha_0} = \theta_\alpha + \beta_\alpha \mu_{x_\alpha}(1 - \kappa_{x_\alpha x_\alpha})$, $\nu_{\alpha_1} = \kappa_{x_\alpha x_\alpha} \beta_\alpha$ and $\kappa_{x_\alpha x_\alpha} = \sigma_{x_\alpha x_\alpha}(\sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha})^{-1}$ (α^{th} reliability ratio). Clearly,

$$\begin{pmatrix} \bar{Y}^0_\alpha \\ \bar{X}^0_\alpha \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_{x_\alpha} \end{pmatrix} : \frac{1}{n_\alpha} \mathbf{\Sigma}_{\alpha\alpha} \right\}$$

and

$$\mathbf{S}_{\alpha\alpha} = \begin{pmatrix} S_{Y^0_\alpha Y^0_\alpha} & S_{X^0_\alpha Y^0_\alpha} \\ S_{X^0_\alpha Y^0_\alpha} & S_{X^0_\alpha X^0_\alpha} \end{pmatrix} \sim W_2\big(\mathbf{\Sigma}_{\alpha\alpha}, n_\alpha - 1\big),$$

where $\bar{Y}^0_\alpha = (1/n_\alpha) \sum_{j=1}^{n_\alpha} Y^0_{\alpha_j}$, $\bar{X}^0_\alpha = (1/n_\alpha) \sum_{j=1}^{n_\alpha} X^0_{\alpha_j}$ $S_{X^0_\alpha X^0_\alpha} = \sum_{j=1}^{n_\alpha} (X^0_{\alpha_j} - \bar{X}^0_\alpha)^2$, $S_{Y^0_\alpha Y^0_\alpha} = \sum_{j=1}^{n_\alpha} (Y^0_{\alpha_j} - \bar{Y}^0_\alpha)^2$ and $S_{X^0_\alpha Y^0_\alpha} = \sum_{j=1}^{n_\alpha} (X^0_{\alpha_j} - \bar{X}^0_\alpha)(Y^0_{\alpha_j} - \bar{Y}^0_\alpha)$. Here $W_2(\mathbf{\Sigma}_{\alpha\alpha}, n_\alpha - 1)$ stands for the Wishart distribution which is independent of the distribution of $(\bar{Y}^0_\alpha, \bar{X}^0_\alpha)'$. Thus we have $E(\mathbf{S}_{\alpha\alpha}/(n_\alpha - 1)) = \mathbf{\Sigma}_{\alpha\alpha}$ which means that the unbiased estimators of the element of $\mathbf{\Sigma}_{\alpha\alpha}$ are the corresponding elements of $\mathbf{S}_{\alpha\alpha}/(n_\alpha - 1)$. Hence, $E(S_{X^0_\alpha X^0_\alpha}/(n_\alpha - 1)) = \sigma_{X^0_\alpha X^0_\alpha}$.

We consider the problem of estimation of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ when it is suspected that the lines may be parallel i.e. $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$, against $H_a : \boldsymbol{\beta} \neq \beta_0 \mathbf{1}_p$ where β_0 is a scalar and $\mathbf{1}_p = (1, \ldots, 1)'-$ a p-tuple of 1's (Kim and Saleh, 2003). Toward this goal, we propose several estimators of $\boldsymbol{\beta}$ namely, (i) the unrestricted estimators (UE), (ii) the preliminary test estimator (PTE), (iii) the Stein-type estimator (SE) and (iv) the Positive-rule Stein-type estimator (PRSE), and study their properties. These types of estimations have been studied by Saleh and Han (1990), Judge and Bock (1978) among others for models without measurement errors. Preliminary test estimators were introduced by Bancroft (1944) and expanded by Saleh and Sen (1978) in a nonparametric set-up. Stein (1956) and James and Stein (1961) introduced the Stein-type estimators and expanded by Saleh and Sen (1978-1986) and Sen and Saleh (1987) in the nonparametric set-up. For the multiple regression model with measurement errors see Fuller (1987) and Cheng and Van Ness (1999) for details and Schneerweiss (1976) on consistency. Kim and Saleh (2003) introduced the preliminary test estimation in a simple linear model with measurement errors and they expended the study in the problem of simultaneous estimation of the regression parameters in a multiple regression model (Kim and Saleh, 2005). Notice that there are 6 parameters of the linear models and there are many different configurations that lead to the same distributions of the observation models and is not identifiable. Therefore, we need to impose conditions or the parameters to overcome the identifiability problem and obtain consistent estimators. For this reason we consider the following conditions: $(i)\,\boldsymbol{\mu}_{x_\alpha} = (\mu_{x_1}, \dots, \mu_{x_p})'$, $(ii)\,\sigma_{x_\alpha x_\alpha} = \sigma_{xx}$, $(iii)\,\sigma_{u_\alpha u_\alpha} = \sigma_{uu}$ and (iv) the reliability ratio, κ_{xx} known for all $\alpha = 1, \dots, p$. We organize the paper as follows. In section 2, we provide the proposed five estimators and motivate the estimators in various ways starting from the unrestricted estimator. Section 3 contains the asymptotic distributional properties of the estimators. The asymptotic analysis of the estimators is carried under a sequence of local alternative against parallelism. In section 4, we obtain the asymptotic distributional bias, quadratic bias, MSE-matrices and risk (under a quadratic loss function) expressions. We provide the comparison of the estimators based on the asymptotic distributional bias, MSE matrix as well as risk analysis. We conclude the paper in section 5.

2 Estimation and Test in Parallelism Model

Consider the model (1.1) and the assumptions that (i) $\boldsymbol{\mu}_x = (\mu_{x_1}, \dots, \mu_{x_p})'$, (ii) $\sigma_{x_\alpha x_\alpha} = \sigma_{xx}$, (iii) $\sigma_{u_\alpha u_\alpha} = \sigma_{uu}$ for all $\alpha = 1, \dots, p$ and (iv) $\kappa_{\alpha\alpha}$ known. Then we notice that

$$\begin{pmatrix} \overline{Y}^0_\alpha \\ \overline{X}^0_\alpha \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_{x_\alpha} \end{pmatrix} : \frac{1}{n_\alpha} \boldsymbol{\Sigma}_{\alpha\alpha} \right\} \tag{2.1}$$

and

$$\mathbf{S}_{\alpha\alpha} = \begin{pmatrix} S_{Y^0_\alpha Y^0_\alpha} & S_{X^0_\alpha Y^0_\alpha} \\ S_{X^0_\alpha Y^0_\alpha} & S_{X^0_\alpha X^0_\alpha} \end{pmatrix} \sim W_2\big(\boldsymbol{\Sigma}_{\alpha\alpha}, n_\alpha - 1\big), \tag{2.2}$$

where

$$\boldsymbol{\Sigma}_{\alpha\alpha} = \begin{pmatrix} \beta_\alpha^2 \sigma_{xx} + \sigma_{ee} & \beta_\alpha \sigma_{xx} \\ \beta_\alpha \sigma_{xx} & \sigma_{xx} + \sigma_{uu} \end{pmatrix}. \tag{2.3}$$

This allows us to estimate $\sigma_{X^0 X^0} = \sigma_{xx} + \sigma_{uu}$ by pooling the $S_{X^0_\alpha X^0_\alpha}$'s, i.e. by $S^0_{X^0 X^0} = \sum_{\alpha=1}^p S_{X^0_\alpha X^0_\alpha}$ since

$$E\Big(\frac{S^0_{X^0 X^0}}{n-p}\Big) = \sigma_{X^0 X^0}. \tag{2.4}$$

Clearly, $S^0_{X^0X^0}/\sigma_{X^0X^0}$ follows the central chi-square distribution with $n-p$ degree of freedom (d.f.). We obtain the unrestricted estimator of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ via the conditional model i.e. $\left(Y^0_{\alpha_j}|X^0_{\alpha_j}\right)$ given by

$$Y^0_{\alpha_j} = \nu_{\alpha_0} + \nu_{\alpha_1}X^0_{\alpha_j} + Z_{\alpha_j}; \;\; \alpha = 1,\dots,p, \tag{2.5}$$

where $Z_{\alpha_j} \sim \mathcal{N}(0, \sigma^{(\alpha)}_{zz})$ where $\sigma^{(\alpha)}_{zz} = \sigma_{ee} + \beta_\alpha\sigma_{xx}(1-\kappa_{xx})$. This model is exactly similar to the standard model except that (i) ν_{α_0} is a translation of θ_α, (ii) ν_{α_1} is a scaled version of β_α and (iii) $\sigma^{(\alpha)}_{zz}$ is the inflated version of σ_{ee}. The estimator of ν_{α_0} and ν_{α_1} conditionally on $(\overline{X}_\alpha, \mathbf{S}_{\alpha\alpha})$ are given by

$$\tilde{\nu}_{\alpha_0} = \overline{Y}_\alpha - \overline{X}_\alpha\tilde{\nu}_{\alpha_1}; \quad \tilde{\nu}_{\alpha_1} = \frac{S_{X^0_\alpha Y^0_\alpha}}{S_{X^0_\alpha X^0_\alpha}}. \tag{2.6}$$

Clearly,

$$\left\{\begin{array}{c} (\tilde{\nu}_{\alpha_0} - \nu_{\alpha_0}) \\ (\tilde{\nu}_{\alpha_1} - \nu_{\alpha_1}) \end{array}\right\} \sim \mathcal{N}_2\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right) : \boldsymbol{\Sigma}^{(\alpha)}\right), \tag{2.7}$$

where

$$\boldsymbol{\Sigma}^{(\alpha)} = \sigma^{(\alpha)}_{zz}\left(\begin{array}{cc} \frac{1}{n_\alpha} + \frac{\overline{X}^{02}_\alpha}{S_{X^0_\alpha X^0_\alpha}} & -\frac{\overline{X}^0_\alpha}{S_{X^0_\alpha X^0_\alpha}} \\ -\frac{\overline{X}^0_\alpha}{S_{X^0_\alpha X^0_\alpha}} & \frac{1}{S_{X^0_\alpha X^0_\alpha}} \end{array}\right). \tag{2.8}$$

Defining $\tilde{\theta}_\alpha = \overline{Y}^0_\alpha - \overline{X}^0_\alpha\tilde{\beta}_\alpha$ and $\tilde{\beta}_\alpha = \kappa^{-1}_{xx}\tilde{\nu}_{\alpha_1}$, we obtain

$$\left\{\begin{array}{c} (\tilde{\theta}_\alpha - \theta_\alpha) \\ (\tilde{\beta}_\alpha - \beta_\alpha) \end{array}\right\} \sim \mathcal{N}_2\left(\left(\begin{array}{c} -(1-\kappa_{xx})(\overline{X}^0_\alpha - \mu_{x_\alpha})\beta_\alpha \\ 0 \end{array}\right) : \boldsymbol{\Sigma}^{(\alpha)}_{\kappa_{xx}}\right),$$

where

$$\boldsymbol{\Sigma}^{(\alpha)}_{\kappa_{xx}} = \sigma^{(\alpha)}_{zz}\left(\begin{array}{cc} \frac{1}{n_\alpha} + \frac{\overline{X}^{02}_\alpha}{\kappa^2_{xx}S_{X^0_\alpha X^0_\alpha}} & -\frac{\overline{X}^0_\alpha}{\kappa^2_{xx}S_{X^0_\alpha X^0_\alpha}} \\ -\frac{\overline{X}^0_\alpha}{\kappa^2_{xx}S_{X^0_\alpha X^0_\alpha}} & \frac{1}{\kappa^2_{xx}S_{X^0_\alpha X^0_\alpha}} \end{array}\right).$$

Further,the unrestricted estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ are given by the LSE/MLE as follows:

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_n = (\tilde{\theta}_1,\dots,\tilde{\theta}_p)' &= \overline{\mathbf{Y}} - \mathbf{T}_{\overline{\mathbf{X}}}\tilde{\boldsymbol{\beta}}_n \\ \tilde{\boldsymbol{\beta}}_n = (\tilde{\beta}_1,\dots,\tilde{\beta}_p)' &= \kappa^{-1}_{xx}\mathbf{S}^{-1}_{X^0X^0}\mathbf{S}_{X^0Y^0}, \end{aligned}$$

where $\overline{\mathbf{Y}} = (\overline{Y}^0_1,\dots,\overline{Y}^0_p)'$; $\mathbf{T}_{\overline{\mathbf{X}}} = \mathrm{Diag}(\overline{X}^0_1,\dots,\overline{X}^0_p)$; $\mathbf{S}_{X^0X^0} = \mathrm{Diag}(S_{X^0_1X^0_1},\dots,S_{X^0_pX^0_p})$; $\mathbf{S}_{X^0Y^0} = (S_{X^0_1Y^0_1},\dots,S_{X^0_pY^0_p})'$. Finally, the unbiased estimator of $\sigma^{(\alpha)}_{zz}$ is given by

$$s^{(\alpha)}_{zz} = (n_\alpha - 2)^{-1}\left|\left|\mathbf{Y}^0_\alpha - \tilde{\nu}_{\alpha_0} - \tilde{\nu}_{\alpha_1}\mathbf{X}^0_\alpha\right|\right|^2. \tag{2.9}$$

The conditional distribution of $\left\{(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta})', (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})'\right\}'$ is given by

$$\begin{pmatrix} \tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p}\left\{ \begin{pmatrix} -(1-\kappa_{xx})(\mathbf{T}_{\overline{\mathbf{X}}} - \mathbf{T}_{\boldsymbol{\mu}})\beta_0 \mathbf{1}_p \\ \mathbf{0} \end{pmatrix}; \mathbf{D} \right\}, \tag{2.10}$$

where

$$\mathbf{T}_{\overline{\mathbf{X}}} = \text{Diag}(\overline{X}_1^0, \ldots, \overline{X}_p^0), \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & -\mathbf{D}_{12} \\ -\mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix},$$
$$\mathbf{D}_{11} = \text{Diag}\Big\{ \Big(\tfrac{1}{n_1} + \tfrac{\overline{X}_1^{02}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}}\Big)\sigma_{zz}^{(1)}, \ldots, \Big(\tfrac{1}{n_p} + \tfrac{\overline{X}_p^{02}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}}\Big)\sigma_{zz}^{(p)} \Big\}$$
$$\mathbf{D}_{22} = \text{Diag}\Big\{ \tfrac{\sigma_{zz}^{(1)}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}}, \ldots, \tfrac{\sigma_{zz}^{(p)}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}} \Big\}, \quad \mathbf{D}_{12} = \mathbf{D}_{21}' = \text{Diag}\Big\{ \tfrac{\overline{X}_1^0 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}}, \ldots, \tfrac{\overline{X}_p^0 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}} \Big\}.$$

In case, the slopes are equal i.e. $\boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ we estimate the common conditional variance $\sigma_{zz}^{(0)} = \sigma_{ee} + \beta_0^2 \sigma_{xx}(1-\kappa_{xx})$ by $s_{zz}^{(0)}$ defined by

$$s_{zz}^{(0)} = (n-2p)^{-1} \sum_{\alpha=1}^{p} (n_\alpha - 2) s_{zz}^{(\alpha)}, \tag{2.11}$$

where $s_{zz}^{(\alpha)}$ is given at (2.9). We define

$$\mathbf{A}_n = \frac{\mathbf{1}_p \mathbf{1}_p' \mathbf{S}_{X^0 X^0}}{\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p} \text{ and } \mathbf{H}_n = \mathbf{I}_p - \mathbf{A}_n.$$

Then we have the restricted estimators (RE) of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ as given by (a) and (b) respectively.

$$(a)\ \hat{\boldsymbol{\theta}}_n = \tilde{\boldsymbol{\theta}}_n + \mathbf{T}_{\overline{\mathbf{X}}} \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \quad \text{and} \quad (b)\ \hat{\boldsymbol{\beta}}_n = \mathbf{A}_n \tilde{\boldsymbol{\beta}}_n.$$

It is easy to obtain the following Theorem.

Theorem 1. *Under $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ and assumed conditions,*

$$\begin{pmatrix} \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p}\left\{ \begin{pmatrix} -(1-\kappa_{xx})(\mathbf{T}_{\overline{\mathbf{X}}} - \mathbf{T}_{\boldsymbol{\mu}})\beta_0 \mathbf{1}_p \\ \mathbf{0} \end{pmatrix}; \sigma_{zz}^{(0)} \begin{pmatrix} \mathbf{D}_{11}^* & \mathbf{D}_{12}^* \\ \mathbf{D}_{21}^* & \mathbf{D}_{22}^* \end{pmatrix} \right\},$$

where

$$\mathbf{D}_{11}^* = \mathbf{N}^{-1} + \frac{\mathbf{T}_{\overline{\mathbf{X}}} \mathbf{1}_p \mathbf{1}_p' \mathbf{T}_{\overline{\mathbf{X}}}}{\kappa_{xx}^2 (\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p)}; \quad \mathbf{D}_{12}^* = \mathbf{D}_{21}^{*'} = -\frac{\mathbf{T}_{\overline{\mathbf{X}}} \mathbf{1}_p \mathbf{1}_p'}{\kappa_{xx}^2 (\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p)}; \quad \mathbf{D}_{22}^* = \frac{1}{\kappa_{xx}^2 \mathbf{1}' \mathbf{S}_{X^0 X^0}^{-1} \mathbf{1}}$$

Test of Parallelism

For the distribution of $\left\{(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta})', (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})\right\}'$ under H_0, we replace $\sigma_{zz}^{(\alpha)}$ by $\sigma_{zz}^{(0)}$ in (2.10) for all $\alpha = 1, \ldots, p$. Now consider the test of the null hypothesis $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$. The conditional likelihood ratio test is given by

$$\mathcal{L}_n = \frac{\kappa_{xx}^2 \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' \mathbf{S}_{X^0X^0} \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n}{(p-1) s_{zz}^{(0)}}.$$

Under H_0, $\mathcal{L}_n$ follows the central F-distribution with $(p-1, n-2p)$ d.f.

To see this, consider the orthogonal matrix $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)$, where $\boldsymbol{\Gamma}_1$ is a $p \times (p-1)$ matrix and $\boldsymbol{\Gamma}_2$ is a $p-$ vector such that $\boldsymbol{\Gamma}_2' \boldsymbol{\Gamma}_1 = \mathbf{0}$ and $\boldsymbol{\Gamma}_1 \boldsymbol{\Gamma}_1' + \boldsymbol{\Gamma}_2 \boldsymbol{\Gamma}_2' = \mathbf{I}_p$ so that $\boldsymbol{\Gamma}_1 \boldsymbol{\Gamma}_1' = \mathbf{I}_p - \boldsymbol{\Gamma}_2 \boldsymbol{\Gamma}_2'$. Further let us choose $\boldsymbol{\Gamma}_2 = \frac{\mathbf{S}_{X^0X^0}^{1/2} \mathbf{1}_p}{\sqrt{\mathbf{1}_p' \mathbf{S}_{X^0X^0} \mathbf{1}_p}}$, then $\boldsymbol{\Gamma}_1 \boldsymbol{\Gamma}_1' = \mathbf{I}_p - \frac{\mathbf{S}_{X^0X^0}^{1/2} \mathbf{1}_p \mathbf{1}_p' \mathbf{S}_{X^0X^0}^{1/2}}{\mathbf{1}_p' \mathbf{S}_{X^0X^0} \mathbf{1}_p}$, which implies that $\mathbf{H}_n = \mathbf{S}_{X^0X^0}^{-1/2} \boldsymbol{\Gamma}_1 \boldsymbol{\Gamma}_1' \mathbf{S}_{X^0X^0}^{1/2}$. Now define the random variable

$$\mathcal{Z} = \frac{\kappa_{xx}}{\sqrt{\sigma_{zz}^{(*)}}} \boldsymbol{\Gamma}_1' \mathbf{S}_{X^0X^0}^{1/2} \tilde{\boldsymbol{\beta}}_n \sim \mathcal{N}\left(\frac{\kappa_{xx}}{\sqrt{\sigma_{zz}^{(*)}}} \boldsymbol{\Gamma}_1' \mathbf{S}_{X^0X^0}^{1/2} \boldsymbol{\beta}, \mathbf{I}_{p-1} \right),$$

where $\sigma_{zz}^{(*)} = \frac{1}{p} \sum_{\alpha=1}^{p} \sigma_{zz}^{(\alpha)}$. Then under a fixed κ_{xx}, $||\mathcal{Z}||^2$ follows the noncentral chi-square distribution with $p-1$ d.f. with noncentrality parameter $\frac{1}{2}\Delta^2$, with $\Delta^2 = \frac{\kappa_{xx}^2 \boldsymbol{\beta}' \mathbf{H}_n' \mathbf{S}_{X^0X^0} \mathbf{H}_n \boldsymbol{\beta}}{\sigma_{zz}^{(*)}}$. Therefore, under $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$, $\mathcal{L}_n$ follows the central F-distribution with $(p-1, n-2p)$ d.f. since $s_{zz}^{(0)} \sim \sigma_{zz}^{(*)} \chi_{n-2p}^2 = \sigma_{zz}^{(0)} \chi_{n-2p}^2$.

Estimators of Slopes

We may now define a class of estimators, $\boldsymbol{\beta}_n^*$ of the form

$$\boldsymbol{\beta}_n^* = \mathbf{A}_n \tilde{\boldsymbol{\beta}}_n + (\mathbf{I}_p - \mathbf{A}_n) \tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n),$$

where $\mathbf{A}_n$ is an idempotent matrix of rank 1 and $g(\mathcal{L}_n)$ is a non-decreasing function of the test-statistic $\mathcal{L}_n$ for testing the $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ against the $H_A; \boldsymbol{\beta} \neq \beta_0 \mathbf{1}_p$. Accordingly, we have the preliminary test estimators of $\boldsymbol{\beta}$ are as follows:

$$\hat{\boldsymbol{\beta}}_n^{PT} = \tilde{\boldsymbol{\beta}}_n - \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n I(\mathcal{L}_n < \mathcal{L}_{n,\alpha}), \tag{a}$$

where $\mathcal{L}_{n,\alpha}$ is the α significance level of the distribution of $\mathcal{L}_n$. To overcome the dependence of the estimators of PTE on ∞, we define the Stein-type estimator:

$$\hat{\boldsymbol{\beta}}_n^{S} = \tilde{\boldsymbol{\beta}}_n - c \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \mathcal{L}_n^{-1}, \tag{b}$$

where $c = (p-3)m/((p-1)(m+2))$. Notice that we have replaced $I(\mathcal{L}_n < \mathcal{L}_{n,\alpha})$ by $c\mathcal{L}_n^{-1}$ in (a) to obtain (b) and impose the condition that $p \geq 4$. The estimator $\hat{\boldsymbol{\beta}}_n^{S}$ may go past the values

$\hat{\boldsymbol{\beta}}_n$. Thus we consider the convex combinations of $\hat{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\beta}}_n^S$ via preliminary test procedure with critical value c. And we obtain the positive-rule Stein-type estimators of $\boldsymbol{\beta}$ respectively as

$$\hat{\boldsymbol{\beta}}_n^{S+} = \hat{\boldsymbol{\beta}}_n I(\mathcal{L}_n < c) + \hat{\boldsymbol{\beta}}_n^S I(\mathcal{L}_n > c) = \hat{\boldsymbol{\beta}}_n + (1 - c\mathcal{L}_n^{-1}) I(\mathcal{L}_n > c)\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n. \tag{c}$$

3 Asymptotic Properties of Estimators under Local Alternatives

In order to study the properties of the five estimators, we need the distribution of the statistic, $\mathcal{L}_n$ under the alternative $H_A : \boldsymbol{\beta} \neq \beta_0 \mathbf{1}_p$. Now, under any alternative, the exact distribution of $\mathcal{L}_n$ depends on the exact distribution of $(n-2p)s_{zz}^{2(0)}$ which is a weighted sum of central chi-square variables with $(n_\alpha - 2)$ d.f. for given α $(= 1, \ldots, p)$, i.e. $(n-2p)s_{zz}^{2(0)} \stackrel{\mathcal{D}}{=} \sum_{\alpha=1}^p \sigma_{zz}^{2(\alpha)} \chi_{n_\alpha - 2}^2$ by (2.11). Even in the conditional situation, the expressions for bias, MSE-matrices and risks are not available due to the lack of the distribution of $\mathcal{L}_n$. Therefore, we take recourse to the asymptotic methods. Let $n = n_1 + \cdots + n_p$ and $\lim_{n\to\infty} \frac{n_\alpha}{n} = \lambda_\alpha (0 < \lambda_\alpha < 1)$ so that $\sum_{\alpha=1}^p \lambda_\alpha = 1$. Thus, we easily find the distribution by moment method following Fuller (1978) that as $n \to \infty$

$$\sqrt{n}\begin{pmatrix} \tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p}\left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{pmatrix} \right\},$$

with

$$\begin{aligned}
\boldsymbol{\Gamma}_{11} &= \mathrm{Diag}\Big\{\lambda_1^{-1}\Big(\sigma_{v_1v_1} + \frac{\mu_{x_1}^2 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}\Big), \ldots, \lambda_p^{-1}\Big(\sigma_{v_pv_p} + \frac{\mu_{x_p}^2 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}\Big)\Big\} \\
\boldsymbol{\Gamma}_{22} &= \mathrm{Diag}\Big\{\lambda_1^{-1}\frac{\sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}, \ldots, \lambda_p^{-1}\frac{\sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}\Big\} \\
\boldsymbol{\Gamma}_{12} &= \boldsymbol{\Gamma}_{21}' = -\mathrm{Diag}\Big\{\lambda_1^{-1}\frac{\mu_{x_1}^2 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}, \ldots, \lambda_p^{-1}\frac{\mu_{x_p}^2 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0X^0}}\Big\},
\end{aligned}$$

where $\sigma_{v_\alpha v_\alpha} = \sigma_{ee} + \beta_\alpha^2 \sigma_{uu}, \ \ \alpha = 1, \ldots, p.$

Further, we consider a class $\{K_{(n)}\}$ of Pitman local alternatives $K_{(n)} : \boldsymbol{\beta}_{(n)} = \beta_0 \mathbf{1}_p + n^{-\frac{1}{2}}\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is a fixed finite vector. Note that when $\boldsymbol{\delta} = \mathbf{0}$ we deal with the null hypothesis H_0. Then we obtain the following theorem.

Theorem 2. *Under* $\{K_{(n)}\}$ *and the assumed conditions, as* $n \to \infty$,

(a) $\sqrt{n}\begin{pmatrix} \tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p}\left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} ; \begin{pmatrix} \boldsymbol{\Gamma}_{11}^* & \boldsymbol{\Gamma}_{12}^* \\ \boldsymbol{\Gamma}_{21}^* & \boldsymbol{\Gamma}_{22}^* \end{pmatrix} \right\}$

with

$$\boldsymbol{\Gamma}_{11}^* = \sigma_{vv}^{(0)} \boldsymbol{\Lambda}_0^{-1} + \sigma_{zz}^{(0)} \frac{\mathbf{T}_{\boldsymbol{\mu}}^2 \boldsymbol{\Lambda}_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0X^0}}; \ \ \boldsymbol{\Gamma}_{12}^* = \boldsymbol{\Gamma}_{21}^* = -\sigma_{zz}^{(0)} \frac{\mathbf{T}_{\boldsymbol{\mu}} \boldsymbol{\Lambda}_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0X^0}}; \ \ \boldsymbol{\Gamma}_{22}^* = \sigma_{zz}^{(0)} \frac{\boldsymbol{\Lambda}_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0X^0}};$$

(b) $\sqrt{n}\begin{pmatrix} \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p}\left\{\begin{pmatrix} \mathbf{T}\boldsymbol{\mu}\mathbf{J}\boldsymbol{\delta} \\ \boldsymbol{\delta} \end{pmatrix}; \begin{pmatrix} \boldsymbol{\Gamma}'_{11} & \boldsymbol{\Gamma}'_{12} \\ \boldsymbol{\Gamma}'_{21} & \boldsymbol{\Gamma}^*_{22} \end{pmatrix}\right\}$

with

$$\boldsymbol{\Gamma}'_{11} = \sigma^{(0)}_{vv}\boldsymbol{\Lambda}_0^{-1} + \sigma^{(0)}_{zz}\frac{\mathbf{T}\boldsymbol{\mu}\mathbf{1}_p\mathbf{1}'_p\mathbf{T}\boldsymbol{\mu}}{\kappa^2_{xx}\sigma_{X^0X^0}}; \quad \boldsymbol{\Gamma}'_{12} = \boldsymbol{\Gamma}'_{21} = -\sigma^{(0)}_{zz}\frac{\mathbf{T}\boldsymbol{\mu}\mathbf{1}_p\mathbf{1}'_p}{\kappa^2_{xx}\sigma_{X^0X^0}}; \quad \boldsymbol{\Gamma}^*_{22} = \sigma^{(0)}_{zz}\frac{\boldsymbol{\Lambda}_0^{-1}}{\kappa^2_{xx}\sigma_{X^0X^0}}.$$

(c) $\mathcal{L}_n = \frac{\kappa^2_{xx}\tilde{\boldsymbol{\beta}}'_n\mathbf{H}'_n\mathbf{S}_{X^0X^0}\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n}{(p-1)s^{(0)}_{zz}}$ *has asymptotically a noncentral chi-squared distribution with* $(p-1)$ *d.f. and non-centrality parameter* $\frac{1}{2}\Delta^2$ *with* $\Delta^2 = \frac{\kappa^2_{xx}\sigma_{X^0X^0}}{\sigma^{(0)}_{zz}}\left(\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\boldsymbol{\delta}\right)$,

where

$$\sigma^{(0)}_{vv} = \sigma_{ee} + \beta_0^2\sigma_{uu}, \ \ \sigma^{(0)}_{zz} = \sigma_{ee} + \beta_0^2\sigma_{xx}(1-\kappa_{xx})$$
$$\mathbf{T}\boldsymbol{\mu} = \mathit{Diag}(\mu_{x_1},\ldots,\mu_{x_p}) = \lim_{n\to\infty}\mathbf{T}_{\overline{\mathbf{X}}}$$
$$\boldsymbol{\Lambda}_0 = \mathit{Diag}(\lambda_1,\ldots,\lambda_p); \lim_{n\to\infty}\mathbf{H}_n = \mathbf{J} \textit{ and } \mathbf{J} = \mathbf{I}_p - \mathbf{1}_p\mathbf{1}'_p\boldsymbol{\Lambda}_0.$$

Proof. For (c)

$$\mathcal{L}_n = \frac{\kappa^2_{xx}\tilde{\boldsymbol{\beta}}'_n\mathbf{H}'_n\mathbf{S}_{X^0X^0}\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n}{(p-1)s^{(0)}_{zz}} = \frac{\kappa^2_{xx}\sigma_{X^0X^0}}{(p-1)\sigma^{(0)}_{zz}}\left(n\tilde{\boldsymbol{\beta}}'_n\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\tilde{\boldsymbol{\beta}}_n\right) + o_p(1).$$

Thus, $\mathcal{L}_n$ follows closely the noncentral chi-square distribution with $p-1$ d.f. as $n\to\infty$ using Sluskey's theorem since $\sigma^{(\alpha)}_{zz} \to \sigma^{(0)}_{zz}, \mathbf{H}_n \to \mathbf{J}$, $\frac{1}{n}S^0_{X^0_\alpha X^0_\alpha} \to \sigma_{X^0X^0}$ and $\frac{1}{n}S_{X^0_\alpha X^0_\alpha} \to \lambda_\alpha\sigma_{X^0X^0}$ in probability as $n\to\infty$. $\square$

Now we consider the asymptotic distributional bias (ADB), quadratic bias (ADQB), MSE matrices (ADMSE) and the quadratic risks (ADQR) of the five estimators. Thus under $\left\{K_{(n)}\right\}$ and the regularity conditions of section 1 we may obtain the various related expressions.

4 Asymptotic Distributional Bias (ADB), Quadratic Bias (ADQB), Mean-square Error Matrix (ADMSE) and Quadratic Risk (ADQR)

In this section, we consider the bias, quadratic bias, mean-square error matrix and quadratic risks of the four estimators defined in section 3. Thus, we may write the bias-vector for the estimator $\boldsymbol{\beta}^*_n$ as

$$\mathbf{b}^* = \lim_{n\to\infty}\sqrt{n}E\left(\boldsymbol{\beta}^*_n - \boldsymbol{\beta}_{(n)}\right) = -\lim_{n\to\infty}\sqrt{n}E\left(\tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n)\right) = -\boldsymbol{\delta}E\left(g(\chi^2_{p+1}(\triangle^2))\right) \quad (4.1)$$

by Appendix B of Judge and Bock (1978) and Saleh (2002) where $\chi^2_\nu(\triangle^2)$ stands for the non-central chi-square variable with ν d.f. and non-centrality parameter $\frac{1}{2}\triangle^2$. To evaluate the properties of the bias we consider the quadratic bias (QB) defined by

$$B = \mathbf{b}^{*\prime}{\boldsymbol{\Gamma}^*_{22}}^{-1}\mathbf{b}^* \quad = \quad \triangle^2\left\{E\left(g(\chi^2_{p+1}(\triangle^2))\right)\right\}^2. \quad (4.2)$$

Thus, QB of $\mathbf{b}^*$ may be studied as a function of $\triangle^2$, the non-centrality parameter of the chi-square distribution. To obtain the mean-square-error (MSE) matrix of the estimator $\boldsymbol{\beta}_n^*$ we have to evaluate

$$\begin{aligned}\mathbf{M}^* &= \lim_{n\to\infty} nE\Big((\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})'\Big) = \lim_{n\to\infty} nE\Big(\mathbf{A}_n(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)})(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)})'\mathbf{A}_n'\Big)\\ &+ \lim_{n\to\infty} nE\Big(\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n\tilde{\boldsymbol{\beta}}_n'\mathbf{H}_n'g^2(\mathcal{L}_n)\Big) - \lim_{n\to\infty} nE\Big(\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n)\boldsymbol{\beta}'\mathbf{H}_n'\Big)\\ &- \lim_{n\to\infty} nE\Big(\mathbf{H}_n\boldsymbol{\beta}\tilde{\boldsymbol{\beta}}_n'\mathbf{H}_n'g(\mathcal{L}_n)\Big) + \lim_{n\to\infty} n\Big(\mathbf{H}_n\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{H}_n'\Big). \qquad (4.3)\end{aligned}$$

Now, using Appendix B of Judge and Bock(1978) and Chapter 2 of Saleh (2002), we get

$$\begin{aligned}&\lim_{n\to\infty} nE\big(\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n)\big) = \mathbf{J}\boldsymbol{\delta}E\big(g(\chi^2_{p+1}(\triangle^2))\big) \quad \text{and}\\ &\lim_{n\to\infty} nE\big(\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n\tilde{\boldsymbol{\beta}}_n'\mathbf{H}_n'g^2(\mathcal{L}_n)\big) = \mathbf{J}\boldsymbol{\Gamma}^*_{22}\mathbf{J}'E\big(g^2(\chi^2_{p+1}(\triangle^2))\big) + \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'E\big(g^2(\chi^2_{p+3}(\triangle^2))\big). \qquad (4.4)\end{aligned}$$

Similarly,

$$\lim_{n\to\infty} nE\big(\mathbf{H}_n\tilde{\boldsymbol{\beta}}_n\tilde{\boldsymbol{\beta}}_n'\mathbf{H}_n'g(\mathcal{L}_n)\big) = \mathbf{J}\boldsymbol{\Gamma}^*_{22}\mathbf{J}'E\big(g(\chi^2_{p+1}(\triangle^2))\big) + \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'E\big(g(\chi^2_{p+3}(\triangle^2))\big). \qquad (4.5)$$

Finally, we substitute the expressions in (4.4) and (4.5) to (4.3) to get

$$\begin{aligned}\mathbf{M}^* = \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathbf{1}_p\mathbf{1}_p' + \mathbf{J}\boldsymbol{\Gamma}^*_{22}\mathbf{J}'E\Big(g^2(\chi^2_{p+1}(\triangle^2))\Big) + \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\Big\{1 - \Big[2E\Big(g(\chi^2_{p+1}(\triangle^2))\Big)\\ - E\Big(g^2(\chi^2_{p+3}(\triangle^2))\Big)\Big]\Big\}. \qquad (4.6)\end{aligned}$$

The quadratic risk under the loss function

$$L\Big(\boldsymbol{\beta}_n^*, \boldsymbol{\beta}\Big) = n\Big(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}\Big)'\mathbf{Q}\Big(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}\Big) \qquad (4.7)$$

with the positive semi-definite matrix $\mathbf{Q}$ is given by

$$\begin{aligned}R^* &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}(\mathbf{1}_p\mathbf{1}_p'\mathbf{Q}) + \mathrm{tr}(\mathbf{J}\boldsymbol{\Gamma}^*_{22}\mathbf{J}'\mathbf{Q})E\Big[g^2(\chi^2_{p+1}(\triangle^2))\Big]\\ &+ \mathrm{tr}(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\mathbf{Q})\Big[1 - \Big\{2E\Big[g(\chi^2_{p+1}(\triangle^2))\Big] - E\Big[g^2(\chi^2_{p+3}(\triangle^2))\Big]\Big\}\Big]. \qquad (4.8)\end{aligned}$$

From the expressions (4.1), (4.2) , (4.6) and (4.8) we obtain the corresponding bias, quadratic bias, MSE-matrix and the quadratic risks of the unrestricted, the preliminary-test, Stein-type and the positive-rule Stein type's estimators as follows.

ADB and ADQB Expressions and Comparisons

We may obtain the asymptotic distributional bias,

$$\begin{aligned}&\zeta_1(\tilde{\boldsymbol{\beta}}_n) = \mathbf{0}, \quad \zeta_2(\hat{\boldsymbol{\beta}}_n) = -\mathbf{J}\boldsymbol{\delta},\\ &\zeta_3(\hat{\boldsymbol{\beta}}_n^{PT}) = -\mathbf{J}\boldsymbol{\delta}H_{p+1}\Big(\chi^2_{p-1}(\alpha) : \Delta^2\Big), \quad \zeta_4(\hat{\boldsymbol{\beta}}_n^{S}) = -(p-3)\mathbf{J}\boldsymbol{\delta}E\Big[\chi^{-2}_{p+1}(\Delta^2)\Big]\\ &\zeta_5(\hat{\boldsymbol{\beta}}_n^{S+}) = -\mathbf{J}\boldsymbol{\delta}\Big\{H_{p+1}(p-3;\Delta^2) + (p-3)E\Big[\chi^{-2}_{p+1}(\Delta^2)I\Big(\chi^2_{p+1}(\Delta^2) > p-3\Big)\Big]\Big\}\end{aligned}$$

and the quadratic bias expressions as follows.

$$B_1(\tilde{\boldsymbol{\beta}}_n) = 0, \quad B_2(\hat{\boldsymbol{\beta}}_n) = \tfrac{\kappa_{xx}^2 \sigma_{X^0X^0}}{\sigma_{zz}^{(0)}}\Big(\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\boldsymbol{\delta}\Big) = \Delta^2$$
$$B_3(\hat{\boldsymbol{\beta}}_n^{PT}) = \Delta^2\Big\{H_{p+1}\Big(\chi^2_{p-1}(\alpha) : \Delta^2\Big)\Big\}^2, \quad B_4(\hat{\boldsymbol{\beta}}_n^{S}) = (p-3)^2\Delta^2\Big\{E\Big[\chi^{-2}_{p+1}(\Delta^2)\Big]\Big\}^2$$
$$B_5(\hat{\boldsymbol{\beta}}_n^{S+}) = \Delta^2\Big\{H_{p+1}(p-3;\Delta^2) + (p-3)E\Big[\chi^{-2}_{p+1}(\Delta^2)I\Big(\chi^2_{p+1}(\Delta^2) > p-3\Big)\Big]\Big\}^2.$$

Clearly, under H_0, all estimators are unbiased. Also, as $n \to \infty$, all estimators except $\tilde{\boldsymbol{\beta}}_n$ has unbounded bias. When Δ^2 is moderate, the biases satisfy the relation

$$B_1(\tilde{\boldsymbol{\beta}}_n) = 0 < B_3(\hat{\boldsymbol{\beta}}_n^{PT}) < B_2(\hat{\boldsymbol{\beta}}_n); \quad B_1(\tilde{\boldsymbol{\beta}}_n) = 0 < B_5(\hat{\boldsymbol{\beta}}_n^{S+}) < B_4(\hat{\boldsymbol{\beta}}_n^{S}).$$

ADMSE and ADQR Expressions and Comparisons

We may obtain the asymptotic MSE and the weighted risk expressions as follows.

$$\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) = \lim_{n\to\infty} E\Big\{n(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})'|K_{(n)}\Big\} = \tfrac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\boldsymbol{\Lambda}_0^{-1}$$
$$\mathbf{M}_2(\hat{\boldsymbol{\beta}}_n) = \lim_{n\to\infty} E\Big\{n(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})'|K_{(n)}\Big\} = \tfrac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathbf{1}_p\mathbf{1}_p' + \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'$$
$$R_1(\tilde{\boldsymbol{\beta}}_n) = \tfrac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}\Big\{\mathbf{Q}\boldsymbol{\Lambda}_0^{-1}\Big\}, \quad R_2(\hat{\boldsymbol{\beta}}_n) = \tfrac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}\Big\{\mathbf{Q}\mathbf{1}_p\mathbf{1}_p'\Big\} + \boldsymbol{\delta}'\mathbf{J}'\mathbf{Q}\mathbf{J}\boldsymbol{\delta}.$$

Risk Analysis

Now consider the risk analysis of the estimators. The risk-difference of the two estimators is given by

$$\begin{aligned} R_1(\tilde{\boldsymbol{\beta}}_n) - R_2(\hat{\boldsymbol{\beta}}_n) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}(\mathbf{Q}\boldsymbol{\Lambda}_0^{-1}) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}(\mathbf{Q}\mathbf{1}_p\mathbf{1}_p') - \boldsymbol{\delta}'\mathbf{J}'\mathbf{Q}\mathbf{J}\boldsymbol{\delta} \\ &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\{\mathrm{tr}(\mathbf{Q}[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p'])\} - \boldsymbol{\delta}'\mathbf{J}'\mathbf{Q}\mathbf{J}\boldsymbol{\delta}. \end{aligned}$$

Hence, $\hat{\boldsymbol{\beta}}_n$ perform better than $\tilde{\boldsymbol{\beta}}_n$ whenever $\Delta^2 \le \{\mathrm{tr}\{\mathbf{Q}[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p']\}/Ch_{min}(\mathbf{Q}\boldsymbol{\Lambda}_0^{-1})\}$ by Courant theorem in Appendix. If $\mathbf{Q} = \{\kappa_{xx}^2\sigma_{X^0X^0}/\sigma_{zz}^{(0)}\}\boldsymbol{\Lambda}_0$, then $\hat{\boldsymbol{\beta}}_n$ performs better than $\tilde{\boldsymbol{\beta}}_n$ whenever $\Delta^2 < p-1$, otherwise $\tilde{\boldsymbol{\beta}}_n$ performs better. The asymptotic risk difference (RRE) of $\hat{\boldsymbol{\beta}}_n$ compared to $\tilde{\boldsymbol{\beta}}_n$ is given by $ARRE(\hat{\boldsymbol{\beta}}_n : \tilde{\boldsymbol{\beta}}_n) = p/(1+\Delta^2)$ which is a decreasing function of Δ^2. At $\Delta^2 = 0$, it attains its maximum efficiency p and drops down to zero as $n \to \infty$. However, the efficiency belongs to the interval $(1,p)$ when $0 < \Delta^2 < p-1$ and outside this interval $\tilde{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n$.

MSE Analysis

In this case consider the matrix-difference

$$\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n) = \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\Big[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p'\Big] - \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'.$$

This MSE-difference is positive definite whenever for a given non-zero vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_p)'$ we have

$$\frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \boldsymbol{\ell}' \Big[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p \mathbf{1}_p' \Big] \boldsymbol{\ell} - \boldsymbol{\ell}' \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' \boldsymbol{\ell} > 0.$$

That is,

$$\ell' \Big[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p \mathbf{1}_p' \Big] \boldsymbol{\ell} \geq \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \boldsymbol{\ell}' \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' \boldsymbol{\ell}.$$

Since $\boldsymbol{\ell}' \boldsymbol{\Lambda}_0^{-1} \boldsymbol{\ell} > 0$, we consider

$$\max_{\boldsymbol{\ell}} \frac{\boldsymbol{\ell}' \mathbf{J} \boldsymbol{\Lambda}_0^{-1} \boldsymbol{\ell}}{\boldsymbol{\ell}' \boldsymbol{\Lambda}_0^{-1} \boldsymbol{\ell}} \geq \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \max_{\boldsymbol{\ell}} \frac{\boldsymbol{\ell}' \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' \boldsymbol{\ell}}{\boldsymbol{\ell}' \boldsymbol{\Lambda}_0^{-1} \boldsymbol{\ell}} \quad \text{or} \quad Ch_{\max}[\mathbf{J}] \geq \Delta^2.$$

Hence, $\hat{\boldsymbol{\beta}}_n$ performs better than $\tilde{\boldsymbol{\beta}}_n$ if $0 \leq \Delta^2 \leq 1$, otherwise $\tilde{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n$. Note that the determinant of $M_2(\hat{\boldsymbol{\beta}}_n)$ is zero so that it is meaningless to have AMRE's of $\hat{\boldsymbol{\beta}}_n$ to other estimators. For the Preliminary-Test estimator,

$$\begin{aligned}
\mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{PT}) &= \lim_{n\to\infty} E\Big\{ n(\hat{\boldsymbol{\beta}}_n^{PT} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}}_n^{PT} - \boldsymbol{\beta})' | K_{(n)} \Big\} \\
&= \frac{\sigma_{zz}^{(0)} \boldsymbol{\Lambda}_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} - \frac{\sigma_{zz}^{(0)} \mathbf{J} \boldsymbol{\Lambda}_0^{-1} \mathbf{J}'}{\kappa_{xx}^2 \sigma_{X^0 X^0}} H_{p+1}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) \\
&\quad + \Big(\mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' \Big) \Big\{ 2H_{p+1}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) - H_{p+3}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) \Big\} \\
R_3(\hat{\boldsymbol{\beta}}_n^{PT}) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathrm{tr}\Big\{ \mathbf{Q} \boldsymbol{\Lambda}_0^{-1} \Big\} \quad - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathrm{tr}\Big\{ \mathbf{Q} \mathbf{J} \boldsymbol{\Lambda}_0^{-1} \mathbf{J}' \Big\} H_{p+1}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) \\
&\quad + \Big(\boldsymbol{\delta}' \mathbf{J}' \mathbf{Q} \mathbf{J} \boldsymbol{\delta} \Big) \Big\{ 2H_{p+1}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) - H_{p+3}\Big(\chi_{p-1}^2(\alpha) : \Delta^2 \Big) \Big\}.
\end{aligned}$$

Risk Analysis

The risk -difference shows that $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\tilde{\boldsymbol{\beta}}_n$ whenever

$$\Delta^2 \leq \frac{\mathrm{tr}\{\mathbf{Q} \mathbf{J} \boldsymbol{\Lambda}_0^{-1} \mathbf{J}'\} H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{C} h_{max}[\mathbf{Q} \boldsymbol{\Lambda}_0^{-1}] \{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\} \tag{4.9}$$

while $\tilde{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$ if

$$\Delta^2 > \frac{\mathrm{tr}\{\mathbf{Q} \mathbf{J} \boldsymbol{\Lambda}_0^{-1} \mathbf{J}'\} H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{Ch_{min}[\mathbf{Q} \boldsymbol{\Lambda}_0^{-1}] \{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}}.$$

If $\mathbf{Q} = \{\kappa_{xx}^2 \sigma_{X^0 X^0} / \sigma_{zz}^{(0)}\} \boldsymbol{\Lambda}_0$, then $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\tilde{\boldsymbol{\beta}}_n$

$$\Delta^2 \leq \frac{(p-1) H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)\}}{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)}$$

otherwise $\tilde{\boldsymbol{\beta}}_n$ is better than $\hat{\boldsymbol{\beta}}_n^{PT}$.

Now consider the risk-difference of $\hat{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\beta}}_n^{PT}$ under H_0,

$$\begin{aligned} R_2(\hat{\boldsymbol{\beta}}_n) - R_3(\hat{\boldsymbol{\beta}}_n^{PT}) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}\sigma_{X^0X^0}} \mathrm{tr}\{(\mathbf{Q}\mathbf{1}_p\mathbf{1}_p') - \mathbf{Q}\boldsymbol{\Lambda}_0^{-1}\} \\ &+ \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}\sigma_{X^0X^0}} \mathrm{tr}(\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}')H_{p+1}(\chi_{p-1}^2(\alpha); 0) \underset{<}{\geq} 0 \end{aligned}$$

accordingly as

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) \underset{<}{\geq} \frac{\mathrm{tr}[\mathbf{Q}(\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p')]}{\mathrm{tr}[\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']}$$

Thus $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\hat{\boldsymbol{\beta}}_n$ when

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) > \frac{\mathrm{tr}[\mathbf{Q}(\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p')]}{\mathrm{tr}[\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']}$$

If $\mathbf{Q} = \{\kappa_{xx}^2\sigma_{X^0X^0}/\sigma_{zz}^{(0)}\}\boldsymbol{\Lambda}_0$, then $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\hat{\boldsymbol{\beta}}_n$ when

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) > 1$$

otherwise $\hat{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$. Notice that the asymptotic risk of $\hat{\boldsymbol{\beta}}_n$ is unbounded while that of $\hat{\boldsymbol{\beta}}_n^{PT}$ is bounded. In general, $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\hat{\boldsymbol{\beta}}_n$ whenever

$$\Delta^2 > \frac{\mathrm{tr}\{\mathbf{Q}[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p' - \mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]\}}{Ch_{max}[\mathbf{Q}\boldsymbol{\Lambda}_0^{-1}]\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}}$$

and $\hat{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$ whenever

$$\Delta^2 \leq \frac{\mathrm{tr}\{\mathbf{Q}[\boldsymbol{\Lambda}_0^{-1} - \mathbf{1}_p\mathbf{1}_p' - \mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]\}}{Ch_{min}[\mathbf{Q}\boldsymbol{\Lambda}_0^{-1}]\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}}$$

If $\mathbf{Q} = \{\kappa_{xx}^2\sigma_{X^0X^0}/\sigma_{zz}^{(0)}\}\boldsymbol{\Lambda}_0$, then $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\hat{\boldsymbol{\beta}}_n$

$$\Delta^2 > \frac{(p-1)[1 - H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]}{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)}$$

otherwise $\hat{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$ for any fixed α.

Now consider the asymptotic risk efficiency of $\hat{\boldsymbol{\beta}}_n^{PT}$ relative to $\tilde{\boldsymbol{\beta}}_n$ when $\mathbf{Q} = \{\kappa_{xx}^2\sigma_{X^0X^0}/\sigma_{zz}^{(0)}\}\boldsymbol{\Lambda}_0$

$$\begin{aligned} ARRE(\hat{\boldsymbol{\beta}}_n^{PT} : \tilde{\boldsymbol{\beta}}_n) &= \Big[1 - \Big(1 - \frac{1}{p}\Big)H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) \\ &+ \frac{1}{p}\Delta^2\Big\{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\Big\}\Big]^{-1} \end{aligned}$$

We note the following properties of $ARRE(\hat{\boldsymbol{\beta}}_n^{PT} : \tilde{\boldsymbol{\beta}}_n)$.

(i) $ARRE(\hat{\boldsymbol{\beta}}_n^{PT} : \tilde{\boldsymbol{\beta}}_n) \gtreqless 1$ according as

$$\Delta^2 \lesseqgtr \frac{(p-1)H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2)}{\{2H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2) - H_{p+3}(\chi^2_{p-1}(\alpha);\Delta^2)\}} \quad (\leq p-1)$$

(ii) The maximum efficiency is attained at $\Delta^2 = 0$ with the value

$$\left\{1 - \left(1 - \frac{1}{p}\right) H_{p+1}(\chi^2_{p-1}(\alpha);0)\right\}^{-1}$$

(iii) $ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n)$ decreases as a function of Δ^2 for fixed $\alpha \in (0,1)$ crossing the 1-line to a minimum, say, at $\Delta^2 = \Delta^2_{\min}$, then monotonically increases towards unity as $\Delta^2 \to \infty$. Since PTE is not an estimator with uniform dominance over $\tilde{\boldsymbol{\beta}}_n$, one may obtain a PTE with minimum guaranteed efficiency, say, E_0 by solving the inequality

$$\max_{\alpha \in \mathcal{A}} \min_{\Delta^2} \text{ARRE}(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n) \geq E_0.$$

(iv) As $\alpha \to 0$, $\hat{\boldsymbol{\beta}}_n^{PT} \to \boldsymbol{\beta}_n$ while $\hat{\boldsymbol{\beta}}_n^{PT} \to \tilde{\boldsymbol{\beta}}_n$ as $\alpha \to 1$.

Next, we consider the risk efficiency of $\hat{\boldsymbol{\beta}}_n^{PT}$ relative to $\hat{\boldsymbol{\beta}}_n$ for $\mathbf{Q} = \{\kappa^2_{xx}\sigma_{X^0X^0}/\sigma^{(0)}_{zz}\}\boldsymbol{\Lambda}_0$ given by

$$\begin{aligned} ARRE(\hat{\boldsymbol{\beta}}_n^{PT} : \hat{\boldsymbol{\beta}}_n)) = p^{-1}(1+\Delta^2)\Big[1 - \Big(1 - \frac{1}{p}\Big) H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2) \\ + \Delta^2 p^{-1}\Big\{2H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2) - H_{p+3}(\chi^2_{p-1}(\alpha);\Delta^2)\Big\}\Big]^{-1}. \end{aligned}$$

We note the following properties of $ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\hat{\boldsymbol{\beta}}_n)$

(i) $ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\hat{\boldsymbol{\beta}}_n) \gtreqless 1$ according as

$$\Delta^2 \leq \frac{(p-1)\{1 - H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2)\}}{\{2H_{p+1}(\chi^2_{p-1}(\alpha);\Delta^2) - H_{p+3}(\chi^2_{p-1}(\alpha);\Delta^2)\}}.$$

(ii) Under H_0, the efficiency is given by

$$p^{-1}\Big[1 - \Big(1 - \frac{1}{p}\Big) H_{p+1}(\chi^2_{p-1}(\alpha);0)\Big]^{-1} \quad (\geq p^{-1})$$

while that of $\hat{\boldsymbol{\beta}}_n^{PT}$ rel $\tilde{\boldsymbol{\beta}}_n$ is given by

$$\Big[1 - \Big(1 - \frac{1}{p}\Big) H_{p+1}(\chi^2_{p-1}(\alpha);0)\Big]^{-1} \quad (\geq 1)$$

Thus, under H_0 $\quad p^{-1} \leq ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\hat{\boldsymbol{\beta}}_n) \leq ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n)$.

(iii) As $\Delta^2 \to \infty$, $ARRE(\hat{\boldsymbol{\beta}}_n^{PT};\hat{\boldsymbol{\beta}}_n) \to \infty$.

(iv) As $\alpha \to 0$, $\hat{\boldsymbol{\beta}}_n^{PT} \to \hat{\boldsymbol{\beta}}_n$ and as $\alpha \to 1$, $\hat{\boldsymbol{\beta}}_n^{PT} \to \tilde{\boldsymbol{\beta}}_n$.

MSE Analysis

The MSE matrix difference is given by

$$\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{PT}) = \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0X^0}} \mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}' H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) \\ - \mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}' \Big\{ 2H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha);\Delta^2) \Big\}.$$

It may be shown by the method in section 5.8.1 that $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\tilde{\boldsymbol{\beta}}_n$ if

$$\Delta^2 < \frac{H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2)}{2H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha);\Delta^2)}$$

otherwise, $\tilde{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$. Similarly, we can show that $\hat{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$ if

$$\Delta^2 < \frac{\{1 - H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2)\}}{\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha);\Delta^2)\}}.$$

Otherwise, $\hat{\boldsymbol{\beta}}_n^{PT}$ performs better than $\hat{\boldsymbol{\beta}}_n$. However, under H_0, it is seen that $\hat{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{PT}$ whenever $\chi_{p-1}^2(\alpha)$ satisfies the inequality $H_{p+1}(\chi_{p-1}^2(\alpha);0) \leq 1$. Thus, none of the estimators dominates the other.

Now, the MSE based efficiency of $\hat{\boldsymbol{\beta}}_n^{PT}$ relative to $\tilde{\boldsymbol{\beta}}_n$ and $\boldsymbol{\beta}_n$ are given respectively by

$$\begin{aligned} AMRE(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n) &= \frac{|\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n)|^{1/p}}{|\mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{PT})|^{1/p}} \\ &= \Big|\mathbf{I}_p - \mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\boldsymbol{\Lambda}_0 H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) \\ &+ \frac{\kappa_{xx}^2\sigma_{X^0X^0}}{\sigma_{zz}^{(0)}} (\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}\boldsymbol{\Lambda}_0)\{2H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha);\Delta^2)\}\Big|^{-1/p} \\ &= \Big\{1 - H_{p+1}(\chi_{p-1}^2(\alpha);\Delta^2)\Big\}^{-(p-1)/p} \\ &\times \Big|1 + \Delta^2\Big\{2H_{p+1,m}(\chi_{p-1}^2(\alpha);\Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha);\Delta^2)\Big\}\Big|^{-1/p}. \end{aligned}$$

Note that

(i) Under H_0,
$$AMRE\Big(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n\Big) = \Big\{1 - H_{p+1}(\chi_{p-1}^2(\alpha);0)\Big\}^{-1}.$$

(ii) Under the alternatives, that is, $\Delta^2 > 0$, $AMRE(\hat{\boldsymbol{\beta}}_n^{PT};\tilde{\boldsymbol{\beta}}_n)$ is decreasing function of Δ^2 for fixed α until $\Delta^2 = \Delta^2_{\min}$, then it increases towards 1 as $\Delta^2 \to \infty$. We cannot say the same thing for MRE$(\hat{\boldsymbol{\beta}}_n^{PT};\hat{\boldsymbol{\beta}}_n)$.

To determine optimum level of significance α to obtain a minimum guaranteed efficiency E_0 we solve the inequality

$$\max_{\alpha} \min_{\Delta^2} AMRE\big(\hat{\boldsymbol{\beta}}_n^{PT}; \tilde{\boldsymbol{\beta}}_n\big) \geq E_0.$$

Some tabular values are given in Table 5.6.1 for various p and n values. For the Stein-type estimator,

$$\begin{aligned}
\mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^S) &= \lim_{n\to\infty} E\Big\{n(\hat{\boldsymbol{\beta}}_n^S - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}}_n^S - \boldsymbol{\beta})'|K_{(n)}\Big\} = \frac{\sigma_{zz}^{(0)}\boldsymbol{\Lambda}_0^{-1}}{\kappa_{xx}^2\sigma_{X^0X^0}} - \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\Big(\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\Big) \\
&\times \Big\{2E\Big[\chi_{p+1}^{-2}(\Delta^2)\Big] - (p-3)E\Big[\chi_{p+1}^{-4}(\Delta^2)\Big]\Big\} + (p-3)(p-1)\Big(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\Big)E\Big[\chi_{p+3}^{-4}(\Delta^2)\Big] \\
R_4(\hat{\boldsymbol{\beta}}_n^S) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}\Big\{\mathbf{Q}\boldsymbol{\Lambda}_0^{-1}\Big\} - \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathrm{tr}\Big[\mathbf{Q}\Big(\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\Big)\Big]\Big\{2E\Big[\chi_{p+1}^{-2}(\Delta^2)\Big] \\
&- (p-3)E\Big[\chi_{p+1}^{-4}(\Delta^2)\Big]\Big\} + (p-3)(p-1)\Big(\boldsymbol{\delta}'\mathbf{J}'\mathbf{Q}\mathbf{J}\boldsymbol{\delta}\Big)E\Big[\chi_{p+3}^{-4}(\Delta^2)\Big].
\end{aligned}$$

Risk Analysis

First note that the asymptotic risk difference

$$\begin{aligned}
R_1(\tilde{\boldsymbol{\beta}}_n) - R_4(\hat{\boldsymbol{\beta}}_n^S) &= \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}\sigma_{X^0X^0}}\Big\{\mathrm{tr}[\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']\Big(2E[\chi_{p+1}^{-2}(\Delta^2)] \\
&- (p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big) + (p+1)\Delta^2 E[\chi_{p+3}^{-4}(\Delta^2)]\Big\} \\
&= \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}\sigma_{X^0X^0}}\mathrm{tr}[\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']\Big\{(p-3)E[\chi_{p+1}^{-2}(\Delta^2)] \\
&+ 2\Delta^2 E[\chi_{p+3}^{-4}(\Delta^2)]\Big(1 - \frac{(p+1)\boldsymbol{\delta}'\mathbf{J}'\mathbf{Q}\mathbf{J}\boldsymbol{\delta}}{2\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\boldsymbol{\delta}}\mathrm{tr}[\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']\Big)\Big\} \geq 0
\end{aligned}$$

for all Δ^2 and $\mathbf{Q}$ matrices which belong to the set

$$\mathcal{Q} = \Big\{\mathbf{Q} : \frac{\mathrm{tr}(\mathbf{Q}\mathbf{J}\boldsymbol{\Lambda}_0\mathbf{J}')}{Ch_{max}(\mathbf{Q}\boldsymbol{\Lambda}_0^{-1})} \geq \frac{p+1}{2}\Big\}.$$

Hence $\hat{\boldsymbol{\beta}}_n^S$ asymptotically dominates $\tilde{\boldsymbol{\beta}}_n$ uniformly for all $\mathbf{Q} \in \mathcal{Q}$.

MSE Analysis

First consider the comparison of $\tilde{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\beta}}_n^S$. In this case, the MSE matrix-difference is given by

$$\begin{aligned}
\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^S) &= \frac{(p-1)\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\Big\{2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\} \\
&- (p-3)(p-1)(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}')E[\chi_{p+3}^{-4}(\Delta^2)]. \qquad (4.10)
\end{aligned}$$

In order that $\hat{\boldsymbol{\beta}}_n^S$ dominates $\tilde{\boldsymbol{\beta}}_n$ based on MSE matrices, we must show (4.10) is non-negative definite. Thus, consider the quadratic form

$$\begin{aligned}
&\frac{(p-1)\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\boldsymbol{\ell}'\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\boldsymbol{\ell}\Big\{2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\}\\
&\geq\ (p-3)(p-1)\boldsymbol{\ell}'(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}')\boldsymbol{\ell}E[\chi_{p+3}^{-4}(\Delta^2)]
\end{aligned}$$

for a given non-zero vector $\boldsymbol{\ell}=(\ell_1,\ldots,\ell_p)'$. Thus, dividing by $\boldsymbol{\ell}'\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\ell}(>0)$, and maximizing over all ℓ, we have

$$\begin{aligned}
\max_{\boldsymbol{\ell}} \quad & \left\{\frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\frac{\boldsymbol{\ell}'\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\ell}}\Big(2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big)\right\}\\
\geq \quad & (p-1)\max_{\boldsymbol{\ell}}\frac{\boldsymbol{\ell}'\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\ell}}E[\chi_{p+3}^{-4}(\Delta^2)]\\
\Leftrightarrow \quad & \Big\{2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\}\\
\geq \quad & (p-1)\Delta^2E[\chi_{p+3}^{-4}(\Delta^2)]\\
\Leftrightarrow \quad & E[\chi_{p+1}^{-2}(\Delta^2)]\geq(p-2)\Delta^2E[\chi_{p+3}^{-4}(\Delta^2)],
\end{aligned}$$

which is contradiction. $\hat{\boldsymbol{\beta}}_n^S$ doesn't dominate uniformly with respect to the MSE criterion.

The MSE-efficiency of $\hat{\boldsymbol{\beta}}_n^S$ compared to $\tilde{\boldsymbol{\beta}}_n$ is given by

$$\begin{aligned}
AMRE\left(\hat{\boldsymbol{\beta}}_n^S;\tilde{\boldsymbol{\beta}}_n\right) &= \frac{|\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n)|^{1/p}}{|\mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^S)|^{1/p}}\\
&= \Big|\mathbf{I}_p-(p-3)\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\boldsymbol{\Lambda}_0\Big\{2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\}\\
&\qquad +(p-1)(p-3)\frac{\kappa_{xx}^2\sigma_{X^0X^0}}{\sigma_{zz}^0}(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}\boldsymbol{\Lambda}_0)E[\chi_{p+3}^{-4}(\Delta^2)]\Big|^{-1/p}\\
&= \Big[1-(p-1)(p-3)\Big\{2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\}\Delta^2\Big]^{-1/p}\\
&\times \Big[1-(p-3)\Big\{2E[\chi_{p+1}^{-2}(\Delta^2)]-(p-3)E[\chi_{p+1}^{-4}(\Delta^2)]\Big\}\Big]^{-(p-1)/p}.
\end{aligned}$$

Under H_0, the $AMRE(\hat{\boldsymbol{\beta}}_n^S;\tilde{\boldsymbol{\beta}}_n)$ becomes

$$AMRE\left(\hat{\boldsymbol{\beta}}_n^S;\tilde{\boldsymbol{\beta}}_n\right)=\left\{1-\frac{p-3}{p-1}\right\}^{-(p-1)/p}(\geq 1),$$

and as $\Delta^2\to\infty$, $AMRE(\hat{\boldsymbol{\beta}}_n^S;\tilde{\boldsymbol{\beta}}_n)\to 1$.

For the Positive-Rule Stein-type estimator,

$$\begin{aligned}
\mathbf{M}_5(\hat{\boldsymbol{\beta}}_n^{S+}) &= \lim_{n\to\infty} E\Big\{n(\hat{\boldsymbol{\beta}}_n^{S+}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}_n^{S+}-\boldsymbol{\beta})'|K_{(n)}\Big\} = \mathbf{M}_4(\hat{\boldsymbol{\theta}}_n^{S}) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\Big\{\Big(\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\Big) \\
&\times\ E\Big[\Big(1-(p-3)\chi_{p+1}^{-2}(\Delta^2)\Big)^2 I\Big(\chi_{p+1}^2(\Delta^2)>p-3\Big)\Big] \\
&+\ \frac{\kappa_{xx}^2\sigma_{X^0X^0}}{\sigma_{zz}^{(0)}}\Big(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\Big)E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)^2\times I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big]\Big\} \\
&-\ 2\Big(\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\Big)\times E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big].
\end{aligned}$$

$$\begin{aligned}
R_5(\hat{\boldsymbol{\beta}}_n^{S+}) &= R_4(\hat{\boldsymbol{\beta}}_n^{S}) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\Big\{\mathrm{tr}\Big(\mathbf{QJ}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'\Big) \\
&\times\ E\Big[\Big(1-(p-3)\chi_{p+1}^{-2}(\Delta^2)\Big)^2 I\Big(\chi_{p+1}^2(\Delta^2)>p-3\Big)\Big] \\
&+\ \frac{\kappa_{xx}^2\sigma_{X^0X^0}}{\sigma_{zz}^{(0)}}\Big(\boldsymbol{\delta}'\mathbf{J}'\mathbf{QJ}\boldsymbol{\delta}\Big)E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)^2\times I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big]\Big\} \\
&-\ 2\Big(\boldsymbol{\delta}'\mathbf{J}'\mathbf{QJ}\boldsymbol{\delta}\Big)\times E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big].
\end{aligned}$$

Risk Analysis

Next we compare $\hat{\boldsymbol{\beta}}_n^{S}$ and $\hat{\boldsymbol{\beta}}_n^{S+}$.

$$\begin{aligned}
R_4(\hat{\boldsymbol{\beta}}_n^{S}) - R_5(\hat{\boldsymbol{\beta}}_n^{S+}) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}\sigma_{X^0X^0}}\Big\{\mathrm{tr}[\mathbf{QJ}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}']E\Big[\Big(1-(p-3)\chi_{p+1}^{-2}(\Delta^2)\Big)^2 I(\chi_{p+1}^2(\Delta^2)>p-3)\Big] \\
&+\ \frac{\kappa_{xx}\sigma_{X^0X^0}}{\sigma_{zz}^{(0)}}(\boldsymbol{\delta}'\mathbf{JQJ}\boldsymbol{\delta})E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)^2 I(\chi_{p+3}^2(\Delta^2)>p-3)\Big]\Big\} \\
&+\ 2(\boldsymbol{\delta}'\mathbf{J}'\mathbf{QJ}\boldsymbol{\delta})E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)I(\chi_{p+3}^2(\Delta^2)>p-3)\Big]\geq 0
\end{aligned}$$

for all $\boldsymbol{\delta}$ and $\mathbf{Q}$. Hence $\hat{\boldsymbol{\beta}}_n^{S+}$ asymptotically dominates $\hat{\boldsymbol{\beta}}_n^{S}$ uniformly. Thus we may order the estimators accordingly to the asymptotic risk as

$$R_5(\hat{\boldsymbol{\beta}}_n^{S+}) < R_4(\hat{\boldsymbol{\beta}}_n^{S}) < R_1(\tilde{\boldsymbol{\beta}}_n)$$

for all $(\Delta^2, \mathbf{Q})$ and $\mathbf{Q}$ satisfying (4.10). It may be verified that $ARRE(\hat{\boldsymbol{\beta}}_n^{S};\tilde{\boldsymbol{\beta}}_n)$ and $ARRE(\hat{\boldsymbol{\beta}}_n^{S+};\hat{\boldsymbol{\beta}}_n^{S})$ are decreasing functions of Δ^2 bounded below by the 1–line.

The risk-efficiency of $\hat{\boldsymbol{\beta}}_n^{S}$ compared to $\tilde{\boldsymbol{\beta}}_n$ for $\mathbf{Q}=\{\kappa_{xx}^2\sigma_{X^0X^0}/\sigma_{zz}^{(0)}\}\boldsymbol{\Lambda}_0$ is given by

$$\begin{aligned}
ARRE(\hat{\boldsymbol{\beta}}_n^{S};\tilde{\boldsymbol{\beta}}_n) &= \Bigg(1-\frac{(p-3)(p-1)}{p}\Big\{2E\Big[\chi_{p+1}^{-2}(\Delta^2)\Big] \\
&\qquad -(p-3)E\Big[\chi_{p+1}^{-4}(\Delta^2)\Big]-\Delta^2 E\Big[\chi_{p+3}^{-4}(\Delta^2)\Big]\Big\}\Bigg)^{-1}.
\end{aligned}$$

Similarly, the risk-efficiency of $\hat{\boldsymbol{\beta}}_n^{S+}$ compared to $\hat{\boldsymbol{\beta}}_n^{S}$ may be written as

$$\begin{aligned} ARRE(\hat{\boldsymbol{\beta}}_n^{S+};\hat{\boldsymbol{\beta}}_n^{S}) \quad = \quad & \Bigg[1-\{R_3(\hat{\boldsymbol{\beta}}_n^{S})\}^{-1}\frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\Big\{(p-1) \\ & \times E\Big[\Big(1-(p-3)\chi_{p+1}^{-2}(\Delta^2)\Big)^2 I\Big(\chi_{p+1}^2(\Delta^2)>p-3\Big) \\ & +\frac{\kappa_{xx}^2\sigma_{X^0X^0}}{\sigma_{zz}^{(0)}}(\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\boldsymbol{\delta})E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)^2 \\ & \times I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)-2\{R_3(\hat{\boldsymbol{\beta}}_n^{S})\}^{-1}(\boldsymbol{\delta}'\mathbf{J}'\boldsymbol{\Lambda}_0\mathbf{J}\boldsymbol{\delta}) \\ & \times E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big). \end{aligned}$$

MSE Analysis

The MSE-efficiency of $\hat{\boldsymbol{\beta}}_n^{S+}$ compared to $\hat{\boldsymbol{\beta}}_n^{S}$ may be written as

$$\begin{aligned} AMRE\left(\hat{\boldsymbol{\beta}}_n^{S+};\hat{\boldsymbol{\beta}}_n^{S}\right) &= \frac{|\mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^{S})|^{1/p}}{|\mathbf{M}_5(\hat{\boldsymbol{\beta}}_n^{S+})|^{1/p}} \\ &= \Bigg|\mathbf{I}_p-\Big(\mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^{S})\Big)^{-1}\Big\{\frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2\sigma_{X^0X^0}}\mathbf{J}\boldsymbol{\Lambda}_0^{-1}\mathbf{J}'E\Big[\Big(1-(p-3)\chi_{p+1}^{-2}(\Delta^2)\Big)^2 \\ &\times I\Big(\chi_{p+1}^2(\Delta^2)>p-3\Big)\Big]-\mathbf{J}\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{J}'\Big\{E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)^2 \\ &\times I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big]+2E\Big[\Big(1-(p-3)\chi_{p+3}^{-2}(\Delta^2)\Big)I\Big(\chi_{p+3}^2(\Delta^2)>p-3\Big)\Big]\Big\}\Big\}\Bigg|^{-1/p}. \end{aligned}$$

It may be shown that under H_0, the $AMRE(\hat{\boldsymbol{\beta}}_n^{S+};\hat{\boldsymbol{\beta}}_n^{S})\geq 1$ and as $\Delta^2\to\infty$, $AMRE(\hat{\boldsymbol{\beta}}_n^{S+};\hat{\boldsymbol{\beta}}_n^{S})$ tends to 1 from above.

5 Conclusion

It is shown that the positive-rule Stein estimator dominates the usual Stein-type estimator which in turn dominate the unrestricted estimator uniformly for the number of lines more than three. Neither the restricted estimator nor the preliminary-test estimator dominate each other uniformly. Similarly, neither the preliminary-test estimator nor Stein-type estimator (the positive-rule Stein-type estimator) dominate each other uniformly. However, when $p \geq 3$, the positive-rule Stein-type estimator is the preferred choice for application otherwise the unrestricted estimator preferable when $p \leq 2$.

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