A BIVARIATE VERSION OF THE HYPER-POISSON DISTRIBUTION AND SOME OF ITS PROPERTIES

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SUMMARY
A bivariate version of the hyper-Poisson distribution is introduced here through its probability generating function (pgf). We study some of its important aspects by deriving its probability mass function, factorial moments, marginal and conditional distributions and obtain certain recurrence relations for its probabilities, raw moments and factorial moments. Further, the method of maximum likelihood is discussed and the procedures are illustrated using a real life data set.

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1 Introduction

Bivariate discrete distributions have received a great deal of attention in the literature. For details see Kumar (2008), Kocherlakota and Kocherlakota (1992) and references therein. Bardwell and Crow (1964) studied the hyper-Poisson distribution (HP distribution), which they defined as follows. A random variable $X$ is said to follow an HP distribution if it has the following probability mass function (pmf), for $x = 0, 1, \ldots$

$$g(x) = P(X = x) = \frac{\theta^x \Gamma(\lambda)}{\phi(1; \lambda; \theta) \Gamma(\lambda + x)}, \quad (1.1)$$

in which $\lambda, \theta$ are positive real numbers and $\phi(1; \lambda; \theta)$ is the confluent hypergeometric series (for details see Mathai and Saxena, 1973 or Slater, 1960). The probability generating function (pgf) of the HP distribution with pmf (1.1) is the following

$$G(t) = \frac{\phi(1; \lambda; \theta t)}{\phi(1; \lambda; \theta)}, \quad (1.2)$$
which reduces to Poisson distribution when \( \lambda = 1 \) and when \( \lambda \) is a positive integer, the distribution is known as the displaced Poisson distribution studied by Staff (1964). Bardwell and Crow (1964) termed the distribution as sub-Poisson when \( \lambda < 1 \) and super-Poisson when \( \lambda > 1 \). Bardwell and Crow (1964) and Crow and Bardwell (1965) considered various methods of estimation of the parameters of the distribution. Some queuing theory with hyper-Poisson arrivals has been developed by Nisida (1962) and certain results on moments of hyper-Poisson distribution has been studied in Ahmad (1979). Roohi and Ahmad (2003a) discussed the estimation of the parameters of the hyper-Poisson distribution using negative moments. Roohi and Ahmad (2003b) obtained certain recurrence relations for negative moments and ascending factorial moments of the HP distribution. Kemp (2002) developed q-analogue of the HP distribution and Ahmad (2007) introduced and studied recurrence relations for negative moments and ascending factorial moments of the HP distribution.

Kumar and Nair (2011, 2012a, 2012b) introduced modified versions of the HP distribution and discussed some of their applications.

Ahmad (1981) introduced a bivariate version of the HP distribution through the following pgf

\[
Q(t_1, t_2) = (\phi_1 \phi_2)^{-1} \exp[\theta(t_1 - 1)(t_2 - 1)] \phi_1[1; \lambda_1; \theta_1 t_1] \phi_2[1; \lambda_2; \theta_2 t_2],
\]

in which \( \phi_i = \phi(1; \lambda_i; \theta_i) \). For \( r \geq 0, s \geq 0 \), the pmf \( q(r, s) = P(Z_1 = r, Z_2 = s) \) of \( Z = (Z_1, Z_2) \) with pgf (1.3) is the following

\[
q(r, s) = \frac{e^\theta \Gamma(\lambda_1) \Gamma(\lambda_2)}{\phi_1 \phi_2} \sum_{i=0}^{\min(r,s)} \sum_{j=0}^{s-i} \sum_{k=0}^{\lambda} \frac{(-1)^{i+k} \theta_1^{r-i-j} \theta_2^{s-i} \theta_1^{i+j+k}}{\Gamma(\lambda_1 + r - i - j) \Gamma(\lambda_2 + s - i - k)! j! k!},
\]

where \( \lambda_1 > 0, \lambda_2 > 0 \) and \( 0 < \theta \leq \min(\theta_1 / \lambda_1, \theta_2 / \lambda_2) \).

Through the present paper we introduce another bivariate version of the HP distribution, which we named as ‘the bivariate hyper-Poisson distribution (BHPD)’ and obtain its important properties. In section 2, it is shown that the BHPD possesses a random sum structure. Further we obtain its conditional probability distribution, probability mass function and factorial moments in section 2. In section 3, we develop certain recursion formulae for probabilities, raw moments and factorial moments of the BHPD and in section 4 we discuss the estimation of the parameters of the BHPD by the method of maximum likelihood and the distribution has been fitted to a well-known data set and it is observed that the BHPD gives better fit than the bivariate Poisson distribution and the bivariate hyper-Poisson distribution of Ahmad (1981).

Note that the bivariate version of HP distribution introduced in this paper is relatively simple in terms of its pmf and pgf compared to the bivariate version due to Ahmad (1981), and further this bivariate form possess a bivariate random sum structure as given in section 2. The random sum structure arises in several areas of research such as ecology, biology, genetics, physics, operation research etc. For details, see Johnson et al. (2005).

## 2 The BHP distribution

Consider a non-negative integer valued random variable \( X \) following HP distribution with pgf (1.2), in which \( \theta = \theta_1 + \theta_2 + \theta_3, \theta_1 > 0, \theta_2 > 0 \) and \( \theta_3 \geq 0 \). Define \( \alpha_j = \theta_j / \theta, \) for \( j = 1, 2, 3 \) and let

\[
R_{12} = \frac{\theta_1}{\theta_1 + \theta_2},
\]

\[
R_{13} = \frac{\theta_1}{\theta_1 + \theta_3},
\]

\[
R_{23} = \frac{\theta_2}{\theta_1 + \theta_2 + \theta_3}.
\]
\( \{Y_n = (Y_{1n}, Y_{2n}), n = 1, 2, \ldots \} \) be a sequence of independent and identically distributed bivariate Bernoulli random vectors, each with pgf

\[
P(t_1, t_2) = \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2. 
\]

Assume that \( X, Y_1, Y_2, \ldots \) are independent. Let \( T_0 = (T_{10}, T_{20}) = (0, 0) \) and define

\[
T_X = (T_1X, T_2X) = \left( \sum_{x=1}^{X} Y_{1x}, \sum_{x=1}^{X} Y_{2x} \right).
\]

Then the pgf of \( T_X \) is the following, in which \( \Lambda = \phi^{-1}(1; \lambda; \theta_1 + \theta_2 + \theta_3) \).

\[
H(t_1, t_2) = G\{P(t_1, t_2)\} = \Lambda \phi(1; \lambda; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2) 
\quad \text{(2.1)}
\]

We call a distribution with pgf as given in (2.1) as ‘the bivariate hyper-Poisson distribution’ or in short, ‘the BHPD’. Clearly the BHPD with \( \lambda = 1 \) is the bivariate Poisson distribution discussed in Kocherlakotta and Kocherlakotta (1992, pp 90) and when \( \lambda \) is a positive integer the BHPD with pgf (2.1) reduces to the pgf of a bivariate version of the displaced Poisson distribution.

Let \( (X_1, X_2) \) be a random variable having the BHPD with pgf (2.1). Then the marginal pgf of \( X_1 \) and \( X_2 \) are respectively

\[
H_{X_1}(t) = H(t, 1) = \Lambda \phi(1; \lambda; (\theta_1 + \theta_3)t + \theta_2) \quad \text{and} \\
H_{X_2}(t) = H(1, t) = \Lambda \phi(1; \lambda; (\theta_2 + \theta_3)t + \theta_1).
\]

The pgf of \( X_1 + X_2 \) is

\[
H_{X_1+X_2}(t) = H(t, t) = \Lambda \phi(1; \lambda; (\theta_1 + \theta_2)t + \theta_3 t^2),
\]

which is the pgf of a modified version of the HP distribution studied in Kumar and Nair (2011).

Let \( x \) be a non-negative integer such that \( P(X_2 = x) > 0 \). On differentiating (2.1) with respect to \( t_2x \) times and putting \( t_1 = t \) and \( t_2 = 0 \), we get

\[
H^{(0,x)}(t, 0) = (\theta_2 + \theta_3)t^x \left( \prod_{j=0}^{x-1} D_j \right) \Lambda \delta_x(\theta_1 t) 
\quad \text{(2.2)}
\]

where \( D_j = (1 + j)/(\lambda + j) \) and \( \delta_j(t) = \phi(1 + j; \lambda + j; t) \) for \( j = 0, 1, 2, \ldots \).

Now applying the formula for the pgf of the conditional distribution in terms of partial derivatives of the joint pgf, developed by Subrahmaniam (1966), we obtain the conditional pgf of \( X_1 \) given \( X_2 = x \) as

\[
H_{X_1|X_2=x}(t) = \left( \frac{\theta_2 + \theta_3 t}{\theta_2 + \theta_3} \right)^x \phi(1 + x; \lambda + x; \theta_1 t) = H_1(t)H_2(t), 
\quad \text{(2.3)}
\]

where \( H_1(t) \) is the pgf of a binomial random variable with parameters \( x \) and \( p = \theta_3(\theta_2 + \theta_3)^{-1} \) and \( H_2(t) \) is the pgf of a random variable following the HPD with parameters \( 1 + x, \lambda + x \) and
θ. Note that, when θ3 = 0 and/or when x = 0, H1(t) reduces to the pgf of a random variable degenerate at zero. Thus the conditional distribution X1 given X2 = x given in (2.4) can be viewed as the distribution of the sum of independent random variables V1 with pgf H1(t) and V2 with pgf H2(t). Consequently from (2.4) we obtain the following

\[ E(X_1 | X_2 = x) = \frac{x\theta_3}{(\theta_2 + \theta_3)} + \frac{\theta_1 D_x \delta_{x+1}(\theta_1)}{\delta_x(\theta_1)} \]  

(2.4)

\[ \text{Var}(X_1 | X_2 = x) = \frac{x\theta_3^2}{(\theta_2 + \theta_3)^2} + \frac{\theta_1 D_x}{\delta_x^2(\theta_1)} \left\{ D_{x+1}\delta_x(\theta_1)\delta_{x+2}(\theta_1)\theta_1 \right. \\
+ \delta_x(\theta_1)\delta_{x+1}(\theta_1) - D_x[\delta_{x+1}(\theta_1)]^2 \theta_1 \} \]  

(2.5)

In a similar approach, for a non-negative integer x with P(X1 = x) > 0, we can obtain the conditional pgf of X2 given X1 = x by interchanging θ1 and θ2 in (2.3). Therefore it is evident that comments similar to those in case of the conditional distribution of X given X = x are valid regarding conditional distribution of X2 given X1 = x and explicit expressions for E(X2 | X1 = x) and Var(X2 | X1 = x) can be obtained by interchanging θ1 and θ2 in the right hand side expressions of (2.5) and (2.6) respectively.

In order to obtain the probability mass function pmf of the BHPD, we need the following partial derivatives of H(t1, t2), in which r is a non-negative integer.

\[ H^{(r,0)}(t_1, t_2) = (\prod_{i=0}^{r-1} D_i)(\theta_1 + \theta_3 t_2)^r \Lambda \Delta_r(t_1, t_2), \]  

(2.6)

where

\[ \Delta_j(t_1, t_2) = \phi(1 + j; \lambda + j; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2), j = 0, 1, 2, \ldots \]

The following derivatives are needed in the sequel, in which 0 ≤ i ≤ r and j ≥ 1.

\[ \frac{\partial^i (\theta_1 + \theta_3 t_2)^r}{\partial t_2^i} = \frac{r^i \theta_3^i}{(r - i)!} (\theta_1 + \theta_3 t_2)^{r-i} \]  

(2.7)

\[ \frac{\partial^j \Delta_r(t_1, t_2)}{\partial t_2^j} = \prod_{i=r}^{r+j-1} D_i(\theta_2 + \theta_3 t_1)^j \Delta_{r+j}(t_1, t_2). \]  

(2.8)

Differentiating both sides of (2.7) s-times with respect to t2 and applying (2.8) and (2.9), we get the following

\[ H^{(r,s)}(t_1, t_2) = (\prod_{i=0}^{r-1} D_i) \Lambda \sum_{m=0}^{s} \binom{s}{m} \frac{\partial^m (\theta_1 + \theta_3 t_2)^r}{\partial t_2^m} \frac{\partial^{s-m} \Delta_r(t_1, t_2)}{\partial t_2^{s-m}} \]  

\[ = (\prod_{i=0}^{r-1} D_i) \Lambda \sum_{m=0}^{\min(r,s)} \binom{s}{m} \frac{r!}{(r-m)!} \theta_3^m (\theta_1 + \theta_3 t_2)^{r-m} \]  

\[ \times \prod_{i=r}^{r+s-m-1} D_i(\theta_2 + \theta_3 t_1)^{s-m} \Delta_{r+s-m}(t_1, t_2) \]  

(2.9)
Now, by putting \((t_1, t_2) = (0, 0)\) in (2.10) and by dividing \(r!s!\), we get the pmf of the \(BHPD\) as

\[
h(r, s) = \Lambda^r \theta_1^r \theta_2^s \sum_{m=0}^{\min(r,s)} \frac{D^*}{m!(r-m)!(s-m)!} \left( \frac{\theta_3}{\theta_1 \theta_2} \right)^m,
\]

(2.10)

where

\[
D^* = \prod_{j=0}^{r+s-m-1} D_j \quad \text{and} \quad \prod_{j=0}^{k} D_j = 1, \text{for any } k < 0.
\]

By putting \((t_1, t_2) = (1, 1)\) in (2.10) we get the \((r, s)\)th factorial moment \(\mu_{[r,s]}\) of the \(BHPD\) as

\[
\mu_{[r,s]} = \Lambda r!s!(\theta_1 + \theta_3)^r(\theta_2 + \theta_3)^s \sum_{m=0}^{\min(r,s)} \frac{D^* \xi_{r+s-m}}{m!(r-m)!(s-m)!} \beta^m
\]

(2.11)

where \(\xi_j = \phi(1 + j; \lambda + j; \theta_1 + \theta_2 + \theta_3)\), for \(j = 0, 1, \ldots\) and \(\beta = \theta_3(\theta_1 + \theta_3)^{-1}(\theta_2 + \theta_3)^{-1}\).

From (2.12) we have the following, in which \(\psi_j = \Lambda \xi_j\), for \(j = 1, 2, \ldots\)

\[
\begin{align*}
E(X_1) &= \mu_{[1,0]} = D_0 \psi_1(\theta_1 + \theta_3) \quad (2.12) \\
E(X_2) &= \mu_{[0,1]} = D_0 \psi_1(\theta_2 + \theta_3) \quad (2.13) \\
\text{Cov}(X_1, X_2) &= D_0(D_1 \psi_2 - D_0 \psi_1^2)(\theta_1 + \theta_3)(\theta_2 + \theta_3) + D_0 \psi_1 \theta_3 \quad (2.14)
\end{align*}
\]

where \(D_0 \) and \(D_1 \) are as given in (2.2).

### 3 Recurrence relations

Let \((X_1, X_2)\) be a random vector following the \(BHPD\) with pgf (2.1). For \(j=0, 1, 2, \ldots\), define \(\lambda^* + j = (1 + j; \lambda + j)\) and \(\lambda^{(j)}(\lambda + j)^{-1}\). Now, the pmf \(h(r, s)\) of the \(BHPD\) given in (2.11) we denote by \(h(r, s; \lambda^*)\). Then we have the following result in the light of relations:

\[
H(t_1, t_2) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s; \lambda^*)t_1^rt_2^s = \Lambda \phi(1; \lambda; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2)
\]

(3.1)

and

\[
\xi_1 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s; \lambda^* + 1)t_1^rt_2^s = \phi(2; \lambda + 1; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2),
\]

(3.2)

in which \(\xi_1\) is as given in (2.12).

**Result 3.1.** The probability mass function \(h(r, s; \lambda^*)\) of the \(BHPD\) satisfies the following recur-
Then the factorial moment generating function in which

\[ \mu \]

Proof. Result 3.2.

On differentiating both sides of (3.1) with respect to \( t \), we have

\[ H^{(1,0)}(t_1, t_2) = D_0(\theta_1 + \theta_3 t_2)\Lambda \Delta_1(t_1, t_2) \]  \hfill (3.7)

By using (3.2) and (3.8) in (3.7) we get the following, in which \( \psi_1 \) is as given in (2.13).

\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (r+1)h(r+1, s, \lambda^*) t_1^r t_2^s = D_0 \psi_1 [\theta_1 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s, \lambda^* + 1) t_1^r t_2^s + \theta_3 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} h(r, s, \lambda^* + 1) t_1^r t_2^{s+1}] \]  \hfill (3.9)

On equating the coefficient of \( t_1^r t_2^s \) on both sides of (3.9) we get the relation (3.3) and on equating the coefficient of \( t_1^r t_2^s \) on both sides of (3.9) we get the relation (3.4). We omit the proof of relations (3.5) and (3.6) as it is similar to that of relations (3.3) and (3.4).

**Result 3.2.** For \( r, s \geq 0 \), simple recurrence relations for factorial moments \( \mu_{[r,s]}(\lambda^*) \) of order \( (r, s) \) of the \( BHPD \) are the following.

\[
\mu_{[r+1,s]}(\lambda^*) = D_0 \psi_1 (\theta_1 + \theta_3) \mu_{[r,s]}(\lambda^* + 1) + D_0 \psi_1 \theta_3 s \mu_{[r,s-1]}(\lambda^* + 1) \]  \hfill (3.10)

\[
\mu_{[r,s+1]}(\lambda^*) = D_0 \psi_1 (\theta_2 + \theta_3) \mu_{[r,s]}(\lambda^* + 1) + D_0 \psi_1 \theta_3 r \mu_{[r-1,s]}(\lambda^* + 1), \]  \hfill (3.11)

in which \( \mu_{[0,0]}(\lambda^*) = 1. \)

Proof. Let \((X_1, X_2)\) be a random vector having the \( BHPD \) with pgf \( H(t_1, t_2) \) as given in (2.1). Then the factorial moment generating function \( F(t_1, t_2) \) of the \( BHPD \) is

\[
F(t_1, t_2) = H(1 + t_1, 1 + t_2)
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{[r,s]}(\lambda^*) \frac{t_1^r t_2^s}{r! s!}
= \Lambda \phi[1; \lambda; \theta_1 + \theta_2 + \theta_3 + (\theta_1 + \theta_3) t_1 + (\theta_2 + \theta_3) t_2 + \theta_3 t_1 t_2] \]  \hfill (3.12)
Differentiate (3.12) with respect to \( t_1 \) to get
\[
\frac{\partial F(t_1, t_2)}{\partial t_1} = [(\theta_1 + \theta_3) + \theta_3 t_2)] D_0 \Lambda \\
\times \phi[2; \lambda + 1; \theta_1 + \theta_2 + \theta_3 + (\theta_1 + \theta_3)t_1 + (\theta_2 + \theta_3)t_2 + \theta_3 t_1 t_2] \quad (3.13)
\]

Based on the similar argument as in the proof of Result 3.1., by using (3.12) with \( \lambda^* \) replaced by \( \lambda^* + 1 \), one can obtain the following from (3.13).
\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{r+1,s}(\lambda^*) \frac{t_1^{r} t_2^{s}}{r! s!} = D_0 \psi_1((\theta_1 + \theta_3) + \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{r,s}(\lambda^* + 1) \frac{t_1^{r} t_2^{s}}{r! s!}) \\
+ \theta_3 \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{r,s}(\lambda^*) \frac{t_1^{r} t_2^{s+1}}{r! s!} \quad (3.14)
\]

Now on equating the coefficients of \((r! s!)^{-1} t_1^{r} t_2^{s}\) on both sides of (3.14) we obtain the relation (3.10). A similar procedure implies (3.11).

\[\Box\]

**Result 3.3.** Two recurrence relations for the \((r, s)\)th raw moments \( \mu_{r,s}(\lambda^*) \) of the BHPD are:

\[
\mu_{r+1,s}(\lambda^*) = D_0 \psi_1(\theta_1) \sum_{j=0}^{r} \binom{r}{j} \mu_{r-j,s}(\lambda^* + 1) + D_0 \psi_1(\theta_3) \sum_{j=0}^{r} \binom{r}{j} \sum_{k=0}^{s} \binom{s}{k} \mu_{r-j,s-k}(\lambda^* + 1) \quad (3.15)
\]

and

\[
\mu_{r,s+1}(\lambda^*) = D_0 \psi_1(\theta_2) \sum_{k=0}^{s} \binom{s}{k} \mu_{r,s-k}(\lambda^* + 1) + D_0 \psi_1(\theta_3) \sum_{j=0}^{r} \binom{r}{j} \sum_{k=0}^{s} \binom{s}{k} \mu_{r-j,s-k}(\lambda^* + 1) \quad (3.16)
\]

**Proof.** The characteristic function \( A(t_1, t_2) \) of the BHPD with pgf (2.1) is the following. For \((t_1, t_2)\) in \( R^2 \),
\[
A(t_1, t_2) = H(e^{it_1}, e^{it_2}) = \Lambda \phi[1; \lambda^*; \lambda(t_1, t_2; \theta)] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{r,s}(\lambda^*) \frac{(it_1)^r (it_2)^s}{r! s!}, \quad (3.17)
\]

where \( \lambda(t_1, t_2; \theta) = \theta_1 e^{it_1} + \theta_2 e^{it_2} + \theta_3 e^{i(t_1 + t_2)}, \theta = (\theta_1, \theta_2, \theta_3) \) and \( i = \sqrt{-1} \). On differentiating (3.17) with respect to \( t_1 \), we obtain
\[
D_0 \Lambda \phi[2; \lambda^* + 1; \lambda(t_1, t_2; \theta)] \{i(\theta_1 + \theta_3 e^{it_2}) e^{it_1}\} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} i \mu_{r,s}(\lambda^*) \frac{(it_1)^{r-1} (it_2)^s}{(r - 1)! s!}.
\]
By using (3.17) with \( \lambda^* \) replaced by \( \lambda^* + 1 \); and on expanding the exponential functions, we obtain the following, in the light of some standard properties of double sum

\[
\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\mu_{r+1,s}(\lambda^*)(it_1)^r(it_2)^s}{r!s!} = D_0 \psi \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(it_1)^r(it_2)^s}{r!s!} \left\{ \theta_1 \sum_{j=0}^{r} \binom{r}{j} \mu_{r-j,s}(\lambda^* + 1) + \theta_3 \sum_{j=0}^{r} \sum_{k=0}^{s} \binom{s}{k} \mu_{r-j,s-k}(\lambda^* + 1) \right\} \]

(3.18)

Now equate the coefficients of \( (r!s!)^{-1}(it_1)^r(it_2)^s \) on both sides of (3.18) to get the relation (3.15). A similar procedure gives (3.16).

4 Estimation of parameters

Here we obtain the estimators of the BHPD by the method of maximum likelihood. Let \( a(r,s) \) be the observed frequency of the \( (r,s) \)th cell of the bivariate data. Let \( y \) be the highest value of \( r \) observed and \( z \) be the highest value of \( s \) observed. Then by using (2.11) the likelihood function of the sample is the following.

\[
L = \prod_{r=0}^{y} \prod_{s=0}^{z} a(r,s) \Rightarrow \log L = \sum_{r=0}^{y} \sum_{s=0}^{z} a(r,s) \log h(r,s).
\]

Let \( \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3 \) and \( \hat{\lambda} \) denotes the likelihood estimators of \( \theta_1, \theta_2, \theta_3 \) and \( \lambda \) respectively. Now \( \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3 \) and \( \hat{\lambda} \) are obtained by solving the likelihood equations (4.1), (4.2), (4.3) and (4.4) given below.

\[
\frac{\partial \log L}{\partial \theta_1} = 0
\]

Equivalently,

\[
\sum_{r=0}^{y} \sum_{s=0}^{z} a(r,s) \left\{ \frac{-1}{\lambda} \phi(2; \lambda + 1; \theta_1 + \theta_2 + \theta_3) + \frac{\min(r,s)}{(r-m-1)!(s-m-1)!} \frac{D^r \phi_{r-m-1}^{\lambda-m-1} \phi_{s-m-1}^{\lambda-m-1} \psi_m}{\xi(r,s)} \right\} = 0. \quad (4.1)
\]

\[
\frac{\partial \log L}{\partial \theta_2} = 0
\]

Equivalently,

\[
\sum_{r=0}^{y} \sum_{s=0}^{z} a(r,s) \left\{ \frac{\min(r,s)}{(r-m-1)!(s-m-1)!} \frac{D^r \phi_{r-m-1}^{\lambda-m-1} \phi_{s-m-1}^{\lambda-m-1} \psi_m}{\xi(r,s)} \right\} = 0. \quad (4.2)
\]
\[ \frac{\partial \log L}{\partial \theta_3} = 0 \]

Equivalently,

\[ \sum_{r=0}^{y} \sum_{s=0}^{z} a(r,s) \left\{ \frac{-1}{\lambda} \frac{\phi(2; \lambda + 1; \theta_1 + \theta_2 + \theta_3)}{\phi(1; \lambda; \theta_1 + \theta_2 + \theta_3)} + \frac{\min(r,s)}{\xi(r,s)} \frac{D^r \theta_1^{-m} \theta_2^{-m} \theta_3^{-m}}{(r-m)! (s-m)! (m-1)!} \right\} = 0. \] (4.3)

\[ \frac{\partial \log L}{\partial \lambda} = 0 \]

Equivalently,

\[ \sum_{r=0}^{y} \sum_{s=0}^{z} a(r,s) \left\{ \frac{-1}{\phi(1; \lambda; \theta_1 + \theta_2 + \theta_3)} \sum_{x=0}^{\infty} (\theta_1 + \theta_2 + \theta_3)^x \eta(x) + \frac{1}{\xi(r,s)} \sum_{m=0}^{\min(r,s)} \eta(r+s-m) \frac{\theta_1^{-m} \theta_2^{-m} \theta_3^{-m}}{(r-m)! (s-m)! m!} \right\} = 0, \] (4.4)

in which \( \xi(r,s) = \sum_{m=0}^{\min(r,s)} \frac{D^r \theta_1^{-m} \theta_2^{-m} \theta_3^{-m}}{(r-m)! (s-m)! m!} \) and \( \eta(u) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+u)} [\psi(\lambda) - \psi(\lambda + u)] \).

5 An application

Here we illustrate the method of maximum likelihood estimation using a real life data set taken from Patrat (1993). The description of data is as follows: The North Atlantic coastal states in USA can be affected by tropical cyclones. They divided the states into three geographical zones: Zone 1 (Texas, Louisiana, The Mississippi, Alabama), Zone 2 (Florida), and Zone 3 (Other states).

Now the interest is in the study of the joint distribution of the pair \((X_1, X_2)\), where \(X_1\) and \(X_2\) are the yearly frequency of hurricanes affecting respectively zone 1 and zone 3. The observed values of \((X_1, X_2)\) during 93 years from 1899 to 1991 are as given in Table 1. We obtain the corresponding expected frequencies by fitting the bivariate Poisson distribution \((BPD)\), the bivariate hyper-Poisson distribution of Ahmad (1981) \((BHPDA)\) and the bivariate hyper-Poisson distribution \((BHPD)\) introduced in this paper using method of maximum likelihood in Table 1. The estimated values of the parameters of the \(BPD\), the \(BHPD\) and the \(BHPD\) and the chi-square values in respective cases are listed in Table 2. From Table 2, it can be observed that the \(BHPD\) gives a better fit to this data compared to the existing models- the \(BPD\) and the \(BHPD\).
Kumar & Nair

Table 1: Comparison of observed and theoretical frequencies Hurricanes (1899-1991) having affected Zone 1 and Zone 3, using method of maximum likelihood.

<table>
<thead>
<tr>
<th>Zone 1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Zone 3</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
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<td>2</td>
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<td>28.31</td>
<td>12.49</td>
<td>2.95</td>
<td>0.48</td>
</tr>
<tr>
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<td>25.64</td>
<td>14.31</td>
<td>2.50</td>
<td>0.26</td>
<td>42.71</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>2.35</td>
<td>0.42</td>
<td>32.86</td>
</tr>
<tr>
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<tr>
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<td>10.88</td>
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</tr>
<tr>
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<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>2</td>
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<td>3.75</td>
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<td>0.19</td>
<td>12.19</td>
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<td>0.17</td>
</tr>
<tr>
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<td>3.62</td>
<td>0.88</td>
<td>0.12</td>
<td>11.22</td>
</tr>
<tr>
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<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
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<td>1.16</td>
<td>0.32</td>
<td>0.06</td>
<td>3.66</td>
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</tr>
<tr>
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<td>7</td>
<td>2</td>
<td>93</td>
</tr>
<tr>
<td>Total</td>
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<td>27.41</td>
<td>6.49</td>
<td>1.15</td>
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<tr>
<td></td>
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<td>6.52</td>
<td>1.1</td>
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<tr>
<td></td>
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<td>29.53</td>
<td>5.82</td>
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</tr>
</tbody>
</table>
Table 2: Estimated values of the parameters of the $BPD$, the $BHPD_A$ and the $BHPD$ by the method of maximum likelihood estimation and corresponding chi-square values.

<table>
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<tr>
<th>Distributions</th>
<th>Estimation of parameters</th>
<th>Chi−square values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BPD$</td>
<td>$\hat{\theta}_1 = 0.683$, $\hat{\theta}_2 = 0.450$, $\hat{\theta}_3 = 0.021$</td>
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<tr>
<td>$BHPD_A$</td>
<td>$\hat{\theta}_1 = 0.780$, $\hat{\theta}_2 = 0.324$, $\hat{\theta}_3 = 0.021$</td>
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<tr>
<td></td>
<td>$\hat{\lambda}_1 = 1.075$, $\hat{\lambda}_2 = 0.619$</td>
<td></td>
</tr>
<tr>
<td>$BHPD$</td>
<td>$\hat{\theta}_1 = 0.414$, $\hat{\theta}_2 = 0.255$, $\hat{\theta}_3 = 0.049$</td>
<td>0.463</td>
</tr>
<tr>
<td></td>
<td>$\hat{\lambda} = 0.457$</td>
<td></td>
</tr>
</tbody>
</table>

References


