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THE RECURRENCE RELATIONS OF ORDER STATISTICS MOMENTS FOR POWER LOMAX DISTRIBUTION

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SUMMARY

The power Lomax distribution due to Rady et al. (2016) is an alternative to and provides better fits for bladder cancer data (Lee and Wang, 2003) than the Lomax, exponential Lomax, Weibull Lomax, extended Poisson Lomax and beta Lomax distributions. Exact explicit expressions as well as recurrence relations for the single and double (product) moments have been derived from the power Lomax distribution. These recurrence relations enable computation of the mean, variance, skewness and kurtosis of all order statistics for all sample sizes in a simple and efficient manner. By using these relation, the mean, variance, skewness and kurtosis of order statistics for sample sizes up to 5 for various values of shape and scale parameters are tabulated. Finally, remission times (in months) of bladder cancer patients have been analyzed to show how the proposed relations work in practice

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1 Introduction

Order statistics and its functions are of great significance in reliability theory, the life-length of the r-out-of-n system made up of n identical components with independent life-lengths which is the $(n-r+1)th$ order statistic in a sample of size n, $X_{n-r+1:n}$. When $r=1$, it is known as the parallel system. System will function as long as any of the n components survives. However, if $r=n$, it is known as a series system. It has wide applicability in practical problems such as characterization of probability distributions and goodness-of-fit tests,

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entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials; see Arnold et al. (1992), David and Nagaraja (2003), and the references therein for more details.

The usage of moments of order statistics can be especially observed in areas such as quality control testing, reliability theory where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics. In recent times, the moments of order statistics have been tabulated quite extensively for several distributions (Arnold et al., 1992; David, 1981; Joshi, 1978; Balakrishnan and Joshi, 1982; Balakrishnan and Malik, 1985; Balakrishnan et al., 1988; Malik et al., 1988; Mohie El-Din et al., 1991; Kumar et al., 2016; Kumar and Dey, 2017; and Kumar, 2017). Balakrishnan and Malik (1986) established exact and explicit expressions for the means and product moments of order statistics from the linear-exponential distribution; also they established some recurrence relations for both single and product moments of order statistics for the same model. Several papers dealing with characterization of distribution through properties of order statistics are appeared in the literature see Lin (1988), Kamps (1991) and Mohie El-Din et al. (1991).

The recurrence relations of order statistics and its identities are quite useful in reducing the number of operations necessary to obtain a general form for the function as they reduce the amount of direct computation, time and labor. This feature has been well documented in the statistical literature (Arnold and Balakrishnan, 1989). Besides, they are used in characterizing the distributions, which play in integral role for the identification of population distribution from the properties of the sample. The computation of moments of order statistics is a challenging task for many distributions. For this reason, recursive computational methods are often sought.

Rady et al. (2016) introduced three parameter power Lomax (POLO) distribution and obtained some statistical and reliability properties, and also estimated its parameters by maximum likelihood method. A random variable X has the POLO distribution with parameters α, β and λ if its cumulative distribution function (cdf) is

$$F(x;\alpha,\beta,\lambda) = 1 - \Big(\frac{\lambda}{x^{\beta}+\lambda}\Big)^{\alpha},\ x>0,\ \alpha,\beta,\lambda>0 \tag{1.1}$$

and corresponding probability density function (pdf) is

$$f(x;\alpha,\beta,\lambda) = \alpha\beta\lambda^{\alpha}x^{\beta-1}\big(x^{\beta}+\lambda\big)^{-\alpha-1},\ x>0,\ \alpha,\beta,\lambda>0 \tag{1.2}$$

The hazard function of the POLO distribution is given by

$$h(x) = \frac{\alpha\beta x^{\beta-1}}{x^{\beta}+\lambda},\ \ x>0,\alpha,\beta,\lambda>0$$

where α and β are the shape parameter and λ is the scale parameter of the distribution. Note that Lomax distribution is a member of POLO distribution if $\beta = 1$.

In this paper, recurrence relations for all single and product moments of order statistics are derived in a simple recursive manner. The so-obtained relationships enables computation of all the moments of order statistics using some mathematical software (Mathematica, Maple). The rest of the paper is organized as follows. In Section 2, two lemmas are derived for obtaining single moments of order statistics. In Section 3, single and double moments of order statistics are derived. In Section 4, recurrence relations for the single moments and double moments of order statistics are obtained. Tabulations of mean, variance, skewness and kurtosis of order statistics are given in Section 5. The analysis of one real data set has been presented in Section 6. Some concluding remarks are addressed in Section 7.

2 Technical Lemmas

Two technical lemmas are illustrated below:

Lemma 2.1. *Let $F(x)$ and $f(x)$ be given by (1.1) and (1.2), respectively. For $a > 0$, $b > 0$ let*

$$I(a,b) = \int_0^\infty x^a [1 - F(x)]^b f(x)\, dx$$

then

$$I(a,b) = \alpha \lambda^{\frac{a}{\beta}} \frac{\Gamma(\frac{a+\beta}{\beta})\,\Gamma(1 + \alpha(b+1) - \frac{a+\beta}{\beta})}{\Gamma(\alpha(b+1)+1)}.$$

Proof. We can write

$$\begin{aligned} I(a,b) &= \int_0^\infty x^a [1 - F(x)]^b f(x)\, dx \\ &= \int_0^\infty x^a \left[\frac{\lambda}{(x^\beta + \lambda)}\right]^{\alpha b} \alpha\beta\lambda^\alpha x^{\beta-1}(x^\beta + \lambda)^{-\alpha-1}\, dx \\ &= \alpha\beta\lambda^{\alpha(b+1)} \int_0^\infty x^{a+\beta-1}(x^\beta + \lambda)^{-\alpha(b+1)-1}\, dx. \end{aligned} \tag{2.1}$$

The result follows by using equation (3.241) in Gradshteyn and Ryzhik (2014) to calculate the integral in (2.1). The proof is complete.

Lemma 2.2. *Let $F(x)$ and $f(x)$ be given by (1.1) and (1.2), respectively. For $a > 0$, $b > 0$, $p > 0$ and $q > 0$, let*

$$K(p,q,a,b) = \int_0^\infty \int_x^\infty x^p y^q [1 - F(x)]^a [1 - F(y)]^b f(x) f(y)\, dydx$$

then

$$K(p,q,a,b) = \alpha^2 \lambda^{\frac{q+p}{\beta}} \sum_{i=0}^{\frac{q}{\beta}} \binom{\frac{q}{\beta}}{i} \frac{(-1)^i}{c\beta} \times \frac{\Gamma(\frac{p+\beta}{\beta})\Gamma(1 + \alpha(a+c+1) - \frac{p+\beta}{\beta})}{\Gamma(\alpha(a+c+1)+1)},$$

where $c = (1/\beta)[\alpha(b+1) + i - (q/\beta)]$,

Proof. Here

$$\begin{aligned} K(p,q,a,b) &= \int_0^\infty \int_x^\infty x^p y^q [1 - F(x)]^a [1 - F(y)]^b f(x) f(y)\, dydx \\ &= \int_0^\infty x^p [1 - F(x)]^a f(x) I(x)\, dx \end{aligned}$$

where

$$I(x) = \int_x^\infty y^q [1 - F(y)]^b f(y)\, dy = \alpha \int_0^{[\bar{F}(x)]^{\frac{1}{\beta}}} \lambda^{\frac{q}{\beta}} (1-z)^{\frac{q}{\beta}} z^{\alpha(b+1)-1-\frac{q}{\beta}}\, dz.$$

By using binomial expansion, the following expression can be obtained:

$$I(x) = \alpha\lambda^{\frac{q}{\beta}} \sum_{i=0}^{\frac{q}{\beta}} \binom{\frac{q}{\beta}}{i} (-1)^i \int_0^{[\bar{F}(x)]^{\frac{1}{\beta}}} z^{c\beta}\, dz = \alpha\lambda^{\frac{q}{\beta}} \sum_{i=0}^{\frac{q}{\beta}} \binom{\frac{q}{\beta}}{i} \frac{(-1)^i}{c\beta} \left[\bar{F}(x)\right]^c.$$

Hence,

$$K(p,q,a,b) = \alpha\lambda^{\frac{q}{\beta}} \sum_{i=0}^{\frac{q}{\beta}} \binom{\frac{q}{\beta}}{i} \frac{(-1)^i}{c\beta} \int_0^\infty x^p [1 - F(x)]^{(a+c)} f(x)\, dx. \tag{2.2}$$

The result follows by using lemma (2.2) to calculate the integral in (2.2). The proof is complete.

3 Moment of Order Statistics

In this section, the exact explicit forms for the single and double moments of order statistics from the POLO distribution are derived.

3.1 Single Moments

The single moments of order statistics are very important to calculate the variance and draw the inferential techniques for the underlying distribution. In the following, the single moments of order statistics from the POLO distribution are derived. Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics corresponding to $X_1, \ldots, X_n$ from the POLO distribution given in Equation (2) with its cdf in Equation (1). Then pdf of the rth order statistic is

$$f_{X_{(r)}}(x) = C_{r:n} \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r} f(x), \quad x > 0, \quad r = 1, \ldots, n, \tag{3.1}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

Using binomial expansion, (3.1) can be rewritten as

$$f_{X_{(r)}}(x) = C_{r:n} \sum_{\ell=0}^{r-1} (-1)^\ell [1 - F(x)]^{\ell+n-r} f(x) dx.$$

The jth moment of the rth-order statistic $\mu_{r:n}^{(j)} = E\big(X_r^{(j)}\big)$ is given by

$$\mu_{r:n}^{(j)} = C_{r:n} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^\ell \int_0^\infty x^j\, [1 - F(x)]^{n-r+\ell} f(x) dx. \tag{3.2}$$

Using (3.2), Lemma 2.1, the moments of the rth order statistic can be written as:

$$\mu_{r:n}^{(j)} = C_{r:n} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^\ell\, I(j, n-r+\ell). \tag{3.3}$$

The validity of the single moments of order statistics in Equation (3.3) can be checked by using Arnold $et\ al.$ (1992)

$$\sum_{r=1}^{n} \mu_{r:n} = nE(X).$$

In particular, the mean order statistic and the variance of order statistic are

$$\begin{aligned}
\mu_{r:n}^{(1)} &= C_{r:n} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} (-1)^{\ell}\, I(1, n-r+\ell) \\
&= \alpha\lambda^{\frac{1}{\beta}} C_{r:n} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} \frac{(-1)^{\ell}\Gamma(\frac{1+\beta}{\beta})\, \Gamma(1+\alpha(n-r+\ell+1) - \frac{1+\beta}{\beta})}{\Gamma(\alpha(n-r+\ell+1)+1)} \\
\sigma_{r:n}^{(2)} &= \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2 \\
&= \alpha\lambda^{\frac{2}{\beta}} C_{r:n} \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} \frac{(-1)^{\ell}\Gamma(\frac{2+\beta}{\beta})\, \Gamma(1+\alpha(n-r+\ell+1) - \frac{2+\beta}{\beta})}{\Gamma(\alpha(n-r+\ell+1)+1)} - \left[\mu_{r:n}^{(1)}\right]^2.
\end{aligned}$$

Remark 1. If $\beta = 1$ in (3.3), the explicit expression of order statistics for Lomax distribution can be obtained.

3.2 Double Moments

The double moments of order statistics are very important to calculate the variance and draw the inferential techniques for the underlying distribution. In the following, the double moments of order statistics from the POLO distribution are derived.

Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics corresponding to $X_1, \ldots, X_n$ from the POLO distribution given in (1.2) with its cdf in (1.1). The joint pdf of the rth and sth order statistics is

$$f_{X_{(r)},X_{(s)}}(x,y) = C_{r,s:n}\, [F(x)]^{r-1} \big[F(y) - F(x)\big]^{s-1-r} \big[1 - F(y)\big]^{n-s} f(x)f(y) \tag{3.4}$$

for $r,\ s = 1, 2\ldots, n,\ \ r < s, 0 < x < y$, where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Using binomial expansion,(3.4) can be rewritten as

$$\begin{aligned}
f_{X_{(r)},X_{(s)}}(x,y) = C_{r,s:n} &\sum_{\ell_1=0}^{r-1} \sum_{\ell_2=0}^{s-r-1} \binom{r-1}{\ell_1} \binom{s-r-1}{\ell_2} (-1)^{\ell_1+\ell_2} \\
&\times \int_0^{\infty}\int_x^{\infty} x^p y^q [1-F(x)]^{s-r-1-\ell_1+\ell_2} [1-F(y)]^{n-s+\ell_1} f(x)f(y)dydx,
\end{aligned}$$

for $0 < x < y$. Then, the double (product) moments of order statistics

$$\mu_{r,s:n}^{(p,q)} = E\big(X_{r:n}^{(p)} X_{s:n}^{(q)}\big)$$

is given by

$$\begin{aligned}
\mu_{r,s:n}^{(p,q)} = C_{r,s:n} &\sum_{\ell_1=0}^{r-1} \sum_{\ell_2=0}^{s-r-1} \binom{r-1}{\ell_1} \binom{s-r-1}{\ell_2} (-1)^{\ell_1+\ell_2} \\
&\times K\big(p, q, (s-r-1-\ell_1+\ell_2), (n-s+\ell_1)\big).
\end{aligned} \tag{3.5}$$

The validity of the double moments of order statistics in Equation (3.5) can be checked by using Arnold et al. (1992)

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\mu_{r,s:n}=\binom{n}{2}\big[E(X)\big]^2.$$

In particular, the covariance of order statistics is

$$\begin{aligned}\mu_{r,s:n}^{(1,1)}&=C_{r,s:n}\sum_{\ell_1=0}^{r-1}\sum_{\ell_2=0}^{s-r-1}\binom{r-1}{\ell_1}\binom{s-r-1}{\ell_2}(-1)^{\ell_1+\ell_2}\\&\quad\times K(1,1,(s-r-1-\ell_1+\ell_2),(n-s+\ell_1))\\&=\alpha^2\lambda^{\frac{2}{\beta}}C_{r,s:n}\sum_{\ell_1=0}^{r-1}\sum_{\ell_2=0}^{s-r-1}\sum_{i=0}^{\frac{1}{\beta}}\binom{r-1}{\ell_1}\binom{s-r-1}{\ell_2}\frac{(-1)^{\ell_1+\ell_2+i}}{(\alpha(n-s+\ell_1+1)+i-\frac{1}{\beta})}\\&\quad\times\frac{\Gamma(\frac{1+\beta}{\beta})\Gamma(1+\alpha(s-r-\ell_1+\ell_2+\frac{1}{\beta}(\alpha(n-s+\ell_1+1)+i-\frac{1}{\beta}))-\frac{1+\beta}{\beta})}{\Gamma(\alpha(s-r-\ell_1+\ell_2+\frac{1}{\beta}(\alpha(n-s+\ell_1+1)+i-\frac{1}{\beta}))+1)}.\end{aligned}$$

Remark 2. Put $\beta=1$ in (3.5), the explicit expression of product moment of order statistics for Lomax distribution can be obtained.

4 Recurrence Relations of Order Statistics

Here, the recurrence relation for the single and double moments of order statistics from the POLO distribution are derived.

4.1 Recurrence Relation for Single Moments

Theorem 1. *For the distribution given in (1.2) and for* $1\le r\le n$ *and* $j=1,2,\ldots$ *then*

$$\mu_{r:n}^{(j)}=\frac{\alpha\beta(n-r+1)}{\lambda(j+\beta)}\Big(\mu_{r:n}^{(j+\beta)}-\mu_{r-1:n}^{(j+\beta)}\Big)-\frac{1}{\lambda}\mu_{r:n}^{(j+\beta)}.$$

Proof. Clearly (1.1) and (1.2) gives

$$\Big(\frac{x^\beta}{\lambda}+1\Big)f(x)=\frac{\alpha\beta x^{\beta-1}}{\lambda}\bar{F}(x).$$

Therefore, for $j=1,2,\ldots$

$$\begin{aligned}\mu_{r:n}^{(j)}+\frac{1}{\lambda}\mu_{r:n}^{(j+\beta)}&=C_{r:n}\int_0^\infty\Big(x^j+\frac{x^{j+\beta}}{\lambda}\Big)[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx\\&=C_{r:n}\int_0^\infty x^j\Big(1+\frac{x^\beta}{\lambda}\Big)[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx\\&=C_{r:n}\frac{\alpha\beta}{\lambda}\int_0^\infty x^{j+\beta-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}\,dx.\end{aligned}$$

Integrating by parts, the above expression can be written as

$$\mu_{r:n}^{(j)} + \frac{1}{\lambda}\mu_{r:n}^{(j+\beta)} = C_{r:n}\frac{\alpha\beta}{\lambda}\left[\frac{n-r+1}{j+\beta}\int_0^\infty x^{j+\beta}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx\right.$$
$$\left.-\frac{r-1}{j+\beta}\int_0^\infty x^{j+\beta}[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)\,dx\right]$$
$$=\frac{\alpha\beta}{\lambda}\left[\frac{n-r+1}{j+\beta}C_{r:n}\int_0^\infty x^{j+\beta}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx\right.$$
$$\left.-\frac{r-1}{j+i\beta}C_{r:n}\int_0^\infty x^{j+\beta}[F(x)]^{r-2}[1-F(x)]^{n-r+1}f(x)\,dx\right]$$
$$=\frac{\alpha\beta}{\lambda}\left[\left(\frac{n-r+1}{j+\beta}\right)\mu_{r:n}^{(j+\beta)}-\left(\frac{r-1}{j+\beta}\right)\frac{C_{r:n}}{C_{r-1:n}}\mu_{r-1:n}^{(j+\beta)}\right].$$

The result follows.

In particular, upon setting $r=1$ in Theorem 1, the following result can be deduced.

Corollary 4.1. For the POLO distribution given in (1.2),

$$\mu_{1:n}^{(j)} = \frac{1}{\lambda}\mu_{1:n}^{(j+\beta)}\left(\frac{\alpha\beta n}{j+\beta}-1\right). \tag{4.1}$$

4.2 Recurrence Relation for Double Moments

Theorem 2. *For the distribution given in (1.2) and for* $1 \le r < s \le n$ *and* $i, j = 1, 2, \ldots$

$$\mu_{r,s:n}^{(i,j)} = \alpha^2\beta^2\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{(-1)^{i_1+i_2-2}}{\lambda^{(i_1+i_2)}}\left[\frac{n-s+1}{j+i_2\beta}(I_1-I_2)-\frac{s-r-1}{j+i_2\beta}(I_3-I_4)\right],$$

where

$$I_1 = -\frac{n}{i+i_1\beta}\mu_{r-1,s-1;n-1}^{(i+i_1\beta,j+i_2\beta)} + \frac{n}{i+i_1\beta}\mu_{r,s-1;n-1}^{(i+i_1\beta,j+i_2\beta)},$$
$$I_2 = -\frac{r}{i+i_1\beta}\mu_{r,s;n}^{(i+i_1\beta,j+i_2\beta)} + \frac{nr}{i+i_1\beta}\mu_{r+1,s;n}^{(i+i_1\beta,j+i_2\beta)},$$
$$I_3 = -\frac{n}{(s-r-1)(i+i_1\beta)}\mu_{r-1,s-2;n-1}^{(i+i_1\beta,j+i_2\beta)} + \frac{n(n-s+1)}{(s-r-2)(i+i_1\beta)}\mu_{r,s-2;n-1}^{(i+i_1\beta,j+i_2\beta)},$$
$$I_4 = -\frac{r(n-s+1)}{(s-r-1)(i+i_1\beta)}\mu_{r,s-1;n}^{(i+i_1\beta,j+i_2\beta)} + \frac{r(n-s+1)}{(s-r-1)(i+i_1\beta)}\mu_{r+1,s-1;n}^{(i+i_1\beta,j+i_2\beta)}.$$

Proof. Clearly (1.1) and (1.2) gives

$$f(x) = \alpha\beta\sum_{i=0}^{\infty}\frac{(-1)^{i-1}x^{\beta i-1}\bar{F}(x)}{\lambda^i}.$$

Therefore, for $i, j = 1, 2, \dots$

$$\begin{aligned}
\mu_{r,s:n}^{(i,j)} &= \frac{\alpha^2\beta^2 C_{r,s:n}}{\lambda^2}\int_0^\infty\int_x^\infty \frac{x^{i+\beta-1}y^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)\left(1+\frac{y^\beta}{\lambda}\right)} F_{x,y}^\star(r-1, s-r-1, n-s+1, 1)\,dy\,dx \\
&= \alpha^2\beta^2 C_{r,s:n}\sum_{i_1=0}^\infty\sum_{i_2=0}^\infty \frac{(-1)^{i_1+i_2-2}}{\lambda^{(i_1+i_2)}}\int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta-1} \\
&\qquad \times F_{x,y}^\star(r-1, s-r-1, n-s+1, 1)\,dy\,dx \\
&= \alpha^2\beta^2 C_{r,s:n}\sum_{i_1=0}^\infty\sum_{i_2=0}^\infty \frac{(-1)^{i_1+i_2-2}}{\lambda^{(i_1+i_2)}}\int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta-1} \\
&\quad \times \Big[F_{x,y}^\star(r-1, s-r-1, n-s+1, 1) - F_{x,y}^\star(r, s-r-1, n-s+1, 1)\Big]dy\,dx
\end{aligned}$$

where

$$F_{x,y}^\star(k, l, m, r) = [F(x)]^k[F(y)-F(x)]^l[1-F(y)]^m[1-F(x)]^r.$$

Integrating by parts with respect to y, the following expression can be obtained

$$\mu_{r,s:n}^{(i,j)} = \sum_{i_1=0}^\infty\sum_{i_2=0}^\infty \frac{\alpha^2\beta^2(-1)^{i_1+i_2-2}C_{r,s:n}}{\lambda^{(i_1+i_2)}}\left[\frac{(n-s+1)(I_1-I_2)}{(j+i_2\beta)} - \frac{(s-r-1)(I_3-I_4)}{(j+i_2\beta)}\right], \tag{4.2}$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta}F_{x,y}^\star(r-1, s-r-1, n-s, 0)\;dy\,dx \\
I_2 &= \int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta}F_{x,y}^\star(r, s-r-1, n-s, 0)\;dy\,dx \\
I_3 &= \int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta}F_{x,y}^\star(r-1, s-r-2, n-s+1, 0)\;dy\,dx \\
I_4 &= \int_0^\infty\int_x^\infty x^{i+i_1\beta-1}y^{j+i_2\beta}F_{x,y}^\star(r, s-r-2, n-s+1, 0)\;dy\,dx.
\end{aligned}$$

Integrating by parts with respect to x, the following results can be obtained

$$\begin{aligned}
I_1 &= -\frac{(r-1)J_1}{i+i_1\beta} + \frac{(s-r-1)J_2}{i+i_1\beta}, \quad I_2 = -\frac{rJ_3}{i+i_1\beta} + \frac{(s-r-1)J_4}{i+i_1\beta} \\
I_3 &= -\frac{(r-1)J_5}{i+i_1\beta} + \frac{(s-r-2)J_6}{i+i_1\beta}, \quad I_4 = -\frac{rJ_7}{i+i_1\beta} + \frac{(s-r-2)J_8}{i+i_1\beta}
\end{aligned}$$

where

$$J_1 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-2, s-r+1, n-s, 0)}{y^{-(j+i_2\beta)}} dF(x)\, dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r-1,s-1;n-1}}{C_{r-1,s-1:n-1}}$$

$$J_2 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-2, n-s, 0)}{y^{-(j+i_2\beta)}} dF(x)\, dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r,s-1;n-1}}{C_{r,s-1:n-1}}$$

$$J_3 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-1, n-s, 0)}{y^{-(j+i_2\beta)}} dF(x)\, dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r,s;n}}{C_{r,s:n}}$$

$$J_4 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-2, n-s, 0)}{y^{-(j+i_2\beta)}} dF(x) dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r+1,s;n}}{C_{r+1,s:n-1}}$$

$$J_5 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-2, s-r-2, n-s+1, 0)}{y^{-(j+i_2\beta)}} dF(x) dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r-1,s-2;n-1}}{C_{r-1,s-2:n-1}}$$

$$J_6 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-3, n-s+1, 0)}{y^{-(j+i_2\beta)}} dF(x) dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r,s-2;n-1}}{C_{r,s-2,n-1}}$$

$$J_7 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-2, n-s+1, 0)}{y^{-(j+i_2\beta)}} dF(x) dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r,s-1;n}}{C_{r,s-1:n}}$$

$$J_8 = \int_0^\infty \int_x^\infty \frac{x^{i+i_1\beta} F^{\star}_{x,y}(r-1, s-r-3, n-s+1, 0)}{y^{-(j+i_2\beta)}} dF(x) dF(y) = \frac{\mu^{(i+i_1\beta, j+i_2\beta)}_{r+1,s-1;n}}{C_{r+1,s-1:n}}$$

The result follows by combining (10), $(11)-(14)$ and $(15)-(22)$.

In particular, upon setting $s = r+1$ in Theorem 2, the following result can be deduced.

Corollary 4.2. For the POLO distribution given in (1.2) and for $1 \leq r \leq n$

$$\mu^{(i,j)}_{r,r+1:n} = \alpha^2 \beta^2 \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} (-1)^{i_1+i_2-2}\, \lambda^{-(i_1+i_2)} \left(\frac{n-r}{j+i_2\beta} I_1 - \frac{n-r}{j+i_2\beta} I_2 \right), \tag{4.3}$$

where

$$I_1 = -\frac{n}{i+i_1\beta} \mu^{(i+i_1\beta, j+i_2\beta)}_{r-1,r;n-1} + \frac{n}{i+i_1\beta} \mu^{(i+i_1\beta, j+i_2\beta)}_{r,r;n-1},$$
$$I_2 = -\frac{r}{i+i_1\beta} \mu^{(i+i_1\beta, j+i_2\beta)}_{r,r+1;n} + \frac{nr}{i+i_1\beta} \mu^{(i+i_1\beta, j+i_2\beta)}_{r+1,r+1;n}.$$

5 Tabulations of Mean, Variance, Skewness and Kurtosis

The recurrence relations obtained in the preceding sections allow us to evaluate the mean, variance, skewness and kurtosis of all order statistics for all sample sizes in a simple recursive manner. In Tables 2 and 3, the mean values for $\alpha = 0.5(0.5)3$ and $\beta = 5$ and 10 and $\lambda = 1$ and 2 are reported. Tables 2 and 3 show that the mean of order statistics decreases as α increases. In Tables 4 and 5, the variances of order statistics for different values of r, s and n for $\alpha = 0.5(0.5)3$ and $\beta = 5$ and 10 and $\lambda = 1$ and 2 are computed. From Tables 4 and 5, one can observe that as α increases, variances of order statistics decreases. Similar conclusion can be drawn for skewness and kurtosis displayed in Tables 6-9, except some cases. Tabular values of mean, variance, skewness and kurtosis are presented in Appendix. Figures 1-4 present the mean, variance, skewness and kurtosis of the first and the last order statistics for $n = 3, 4$ and 5 for different values of α and $\beta = 5$ and

$\lambda = 1$. From these Figures, one can observe that the mean, variance and skewness of the first and the last order statistics decreases as α increases, except the kurtosis of the last order statistics which increase then decrease as α increases.

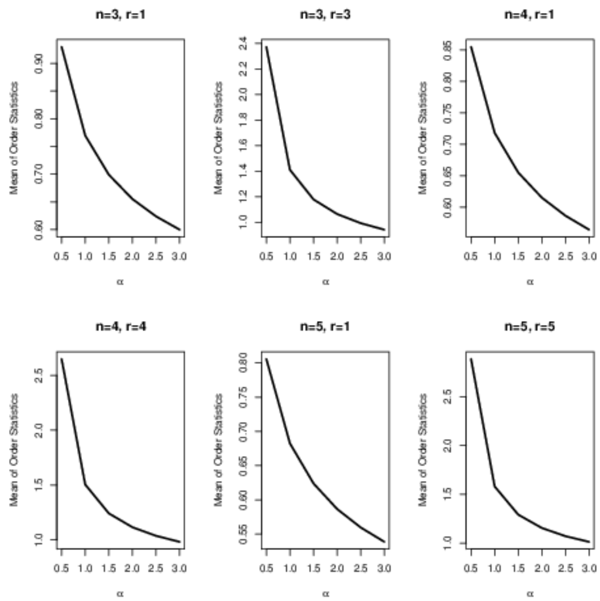

Figure 1: Mean of order statistics for $\beta = 5, \lambda = 1$ and different values of α.

6 Data Analysis

In this section, we analyse a real data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003). Rady et al. (2016) showed that the POLO distribution provides a better fit to this data than Lomax, MCLomax , BLomax and KW Lomax by Lemonte and Cordeiro (2013), exponential Lomax (El-Bassiouny et al., 2015), gamma Lomax (Cordeiro et al., 2013), transmuted exponentiated Lomax (Ashour and Eltehiwy, 2013), Weibull Lomax (Tahir et al., 2015), extended Poisson Lomax (Al-Zahrani, 2015) and exponentiated Lomax (Abdul-Moniem, 2012). They obtained the MLEs of the unknown parameters as: $\hat{\alpha} = 2.070$, $\hat{\beta} = 1.428$ and $\hat{\lambda} = 34.863$.

The above estimates can be used to know how the minimum and maximum remission times (in months) occur on average in every n patients. These remission times can be estimated by $\mu_{1:n}^{(1)}$ and $\mu_{n:n}^{(1)}$, respectively. Table 1 displays the ML predictions of $\mu_{1:n}^{(1)}$ and $\mu_{n:n}^{(1)}$ for $n = 20(20)140$. Also, the values of variance, skewness and kurtosis for these predictions are presented in Table 1. From Table 1, it is to be noted that the values of $\mu_{1:n}^{(1)}$ decreases when the sample size increases while the corresponding variance decreases as the sample size increases. On the other hand, the values of $\mu_{n:n}^{(1)}$ and the corresponding variance increase as the sample size increase, while the corresponding skewness and kurtosis decreases with the increasing in the sample

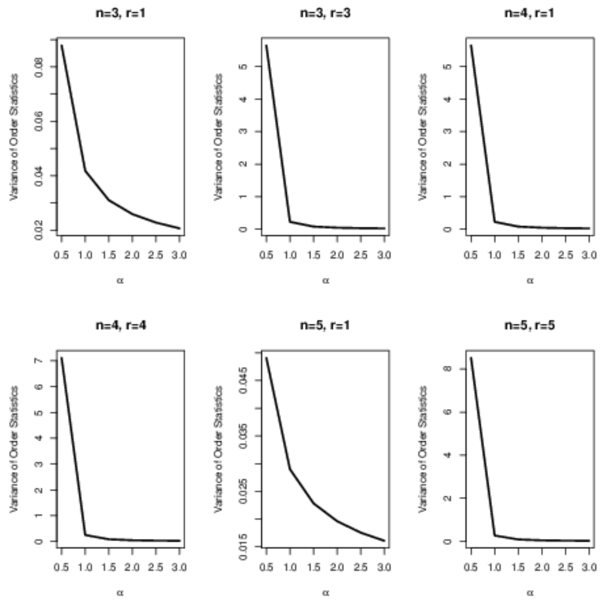

Figure 2: Variance of order statistics for $\beta = 5, \lambda = 1$ and different values of α.

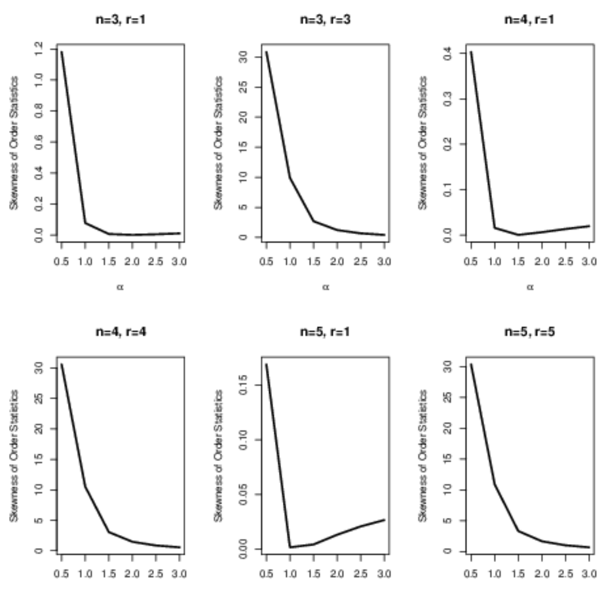

Figure 3: Skewness of order statistics for $\beta = 5, \lambda = 1$ and different values of α.

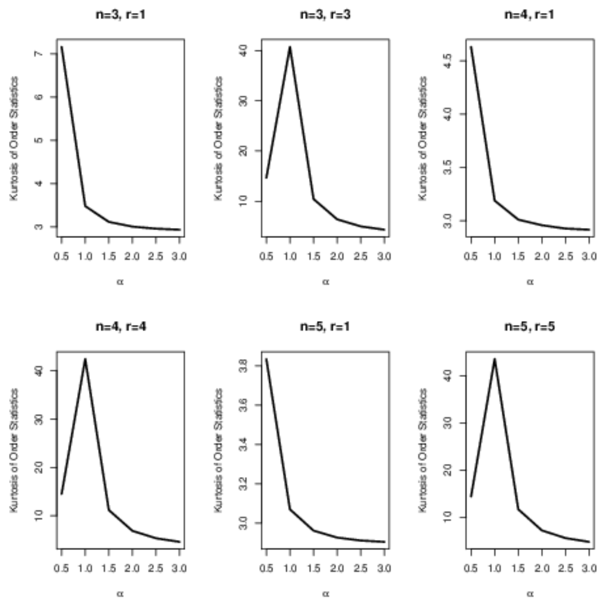

Figure 4: Kurtosis of order statistics for $\beta = 5, \lambda = 1$ and different values of α.

size. Finally, from Table 1, it can be concluded that the minimum expected remission time is 0.21 months and maximum remission time is 83.49 months, respectively.

7 Conclusion

In this paper, recurrence relations for single and product moments of order statistics from POLO distribution have been derived. The recurrence relations for moments of order statistics are important because they can be helpful in reducing the amount of direct calculations needed to calculate the moments, and they can be used in a simple recursive manner to express the unknown higher order moments in terms of order statistics thus making the evaluation of higher moments easy. Also they can be used to characterize the distributions. The work is in progress on inferential issues and it will be reported later.

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Table 1: Estimates of $\mu_{1:n}^{(1)}$ and $\mu_{n:n}^{(1)}$ and the corresponding variance, skewness and kurtosis for real data

	$\mu_{1:n}^{(1)}$				$\mu_{n:n}^{(1)}$			
n	MLE	Variance	Skewness	Kurtosis	MLE	Variance	Skewness	Kurtosis
20	0.817	0.350	1.245	5.111	40.178	1039.429	2.948	22.334
40	0.499	0.128	0.062	0.458	52.581	1643.598	2.327	14.512
60	0.375	0.072	0.317	0.205	61.237	2151.711	1.986	11.014
80	0.306	0.048	1.330	1.440	68.110	2606.208	1.755	8.948
100	0.262	0.035	2.733	4.116	73.906	3024.651	1.583	7.558
120	0.230	0.027	4.050	7.717	78.967	3416.468	1.448	6.549
140	0.207	0.022	4.974	11.364	83.490	3787.462	1.338	5.778

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A Appendix

Table A1: Mean of order statistics for different values of α, β and λ

		$\beta = 5,\ \lambda = 1$			$\beta = 10,\ \lambda = 2$		
n	r	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$
1	1	1.5497	0.9298	0.7697	1.2760	1.0208	0.9316
2	1	1.0689	0.7697	0.6551	1.0896	0.9316	0.8605
	2	2.0304	1.0899	0.8842	1.4624	1.1100	1.0027
3	1	0.9298	0.6990	0.5998	1.0208	0.8885	0.8236
	2	1.3472	0.9109	0.7658	1.2272	1.0178	0.9342
	3	2.3720	1.1795	0.9434	1.5801	1.1562	1.0370
4	1	0.8552	0.6551	0.5643	0.9806	0.8605	0.7990
	2	1.1538	0.8307	0.7063	1.1413	0.9726	0.8975
	3	1.5407	0.9912	0.8251	1.3129	1.0629	0.9709
	4	2.6491	1.2423	0.9828	1.6691	1.1872	1.0590

Table A2: Variance of order statistics for different values of α, β and λ

		$\beta = 5,\ \lambda = 1$			$\beta = 10,\ \lambda = 2$		
n	r	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$
1	1	2.3608	0.0879	0.0419	0.1519	0.0259	0.0162
2	1	0.1786	0.0419	0.0259	0.0407	0.0162	0.0121
	2	4.0807	0.0826	0.0316	0.1936	0.0198	0.0102
3	1	0.0879	0.0310	0.0207	0.0259	0.0135	0.0106
	2	0.2439	0.0336	0.0180	0.0416	0.0105	0.0069
	3	5.6489	0.0831	0.0278	0.2280	0.0182	0.0083
4	1	0.0615	0.0259	0.0179	0.0206	0.0121	0.0098
	2	0.1003	0.0232	0.0139	0.0226	0.0081	0.0058
	3	0.3127	0.0309	0.0150	0.0459	0.0088	0.0052
	4	7.1206	0.0848	0.0259	0.2571	0.0175	0.0073

Table A3: Skewness of order statistics for different values of α, β and λ

		$\beta = 5,\ \lambda = 1$			$\beta = 10,\ \lambda = 2$		
n	r	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$
1	1	30.658	1.1818	0.0768	11.7334	0.1025	0.0437
2	1	6.1766	0.0768	0.0002	0.8776	0.0437	0.1897
	2	31.1178	2.1342	0.2609	13.4439	0.5748	0.0183
3	1	1.1818	0.0059	0.0097	0.1025	0.1304	0.2547
	2	8.9202	0.2536	0.0145	1.9846	0.0172	0.0319
	3	30.8511	2.6949	0.4273	13.5957	0.9108	0.1003
4	1	0.4034	0.0002	0.0193	0.0020	0.1897	0.2913
	2	2.3007	0.0597	0.00	0.6335	0.0054	0.0786
	3	9.8308	0.4111	0.0486	2.4017	0.0934	0.0010
	4	30.5981	3.0554	0.5605	13.5387	1.1386	0.1894

Table A4: Kurtosis of order statistics for different values of α, β and λ

		$\beta = 5,\ \lambda = 1$			$\beta = 10,\ \lambda = 2$		
n	r	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 3$
1	1	14.7921	7.16626	3.4837	48.2997	4.1793	3.4336
2	1	29.5560	3.4837	3.0095	6.5101	3.4336	3.4209
	2	14.8156	9.2913	3.9993	51.7441	5.0312	3.5012
3	1	7.16625	3.1178	2.9378	4.1793	3.3855	3.4513
	2	37.1271	3.9743	3.2038	8.1988	3.4671	3.3094
	3	14.6321	10.4404	4.3370	51.7369	5.5669	3.6625
4	1	4.6305	3.0095	2.9151	3.6337	3.4209	3.4894
	2	9.4917	3.3785	3.0952	5.0530	3.2772	3.2661
	3	39.2805	4.2729	3.3013	8.7362	3.6075	3.2771
	4	14.4867	11.1683	4.5789	51.3831	5.9166	3.6969