Journal of Statistical Research *ISSN 0256 - 422 X*
2018, Vol. 52, No. 2, pp. 173-186

ASYMMETRIC CURVED NORMAL DISTRIBUTION

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SUMMARY

Here we introduce a new class of skew normal distribution as a generalization of the extended skew curved normal distribution of Kumar and Anusree (J. Statist. Res., 2017) and investigate some of its important statistical properties. The location-scale extension of the proposed class of distribution is also defined and discussed the estimation of its parameters by method of maximum likelihood. Further, a real life data set is considered for illustrating the usefulness of the model and a brief simulation study is attempted for assessing the performance of the estimators.

Keywords and phrases: Asymmetric distributions; Maximum likelihood estimation; Model selection; Plurimodality; Simulation

1 Introduction

The literature related to skew-normal distributions has grown rapidly in recent years but at the moment few applications concern the description of natural phenomena with this type of probability models, as well as the interpretation of their parameters. The family of skew-normal distributions represents an extension of the family of normal distribution to which a parameter (λ) has been inserted to regulate the skewness. The skew normal distribution was first introduced by Azzalini (1985) through the following probability density function (p.d.f):

$$g(x;\lambda) = 2\phi(x)\Phi(\lambda x). \tag{1.1}$$

Here $\phi(\cdot)$ and $\Phi(\cdot)$ be the p.d.f and cumulative distribution function (c.d.f) of a standard normal variate and $\lambda \in R = (-\infty, \infty)$ and $x \in R = (-\infty, \infty)$. A distribution with p.d.f. (1.1) hereafter we denoted as $SND(\lambda)$. The $SND(\lambda)$ has been studied by several authors such as Azzalini (1986), Henze (1986), Azzalini and Dalla Valle (1996), Branco and Dey (2001), Kumar and Anusree (2011), Kumar and Anusree (2014a), Kumar and Anusree (2014b) and Kumar and Anila (2018).

Arellano-Valle et al. (2004) introduced a skew-curved normal distribution as follows:

$$g_1(x;\lambda) = 2\phi(x)\Phi\left(\frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}\right). \tag{1.2}$$

in which $x \in R$, $\lambda \in R$. A distribution with pdf (1.2) we denoted as $SCND(\lambda)$.

The $SCND(\lambda)$ of Arellano-Valle et al. (2004) is log concave. Therefore it is not suitable for plurimodal data. To overcome this drawback, Kumar and Anusree (2017) considered an extended version of $SCND(\lambda)$ through the following p.d.f,

$$g_2(x;\lambda,\alpha) = \frac{2}{\alpha+2}\phi(x)\left[1+\alpha\Phi\left(\frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right)\right], \tag{1.3}$$

in which $x \in R$, $\lambda \in R$ and $\alpha \geq -1$. The distribution given in (1.3) they termed as "extended skew curved normal distribution ($ESCND(\lambda,\alpha)$)". Through the present work we propose a modification to the $ESCND(\lambda,\alpha)$ and named it as "Asymmetric curved normal distribution(ACND)". We investigate several important statistical properties of the distribution in Section 2. In section 3 the characteristic function and the expression for the moments are presented. In Section 4 certain reliability measures such as reliability function, mean residual life function are derived and the condition for unimodal and plurimodal situations are obtained. In Section 5 a location scale extension of the ACND is presented and obtained its characteristic function, reliability measures etc. In section 6 maximum likelihood estimation of the parameters of ACND is discussed.The procedure for the generalized likelihood ratio test (GLRT) is discussed in Section 7 and a real life data applications are presented in Section 8. Further a brief simulation study is presented in Section 9.

2 Definition and Properties

Here first we present the definition of the ACND and derive some of its important distributional properties.

Definition 2.1. A random variable X is said to have asymmetric curved normal distribution if its p.d.f is of the following form, in which $x \in R$, $\lambda \in R$, $\beta \in R$ and $\alpha \geq -1$.

$$f(x;\lambda,\alpha,\beta) = \frac{\phi(x)}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right)\right], \tag{2.1}$$

where $\phi(.)$ and $\Phi(.)$ are the p.d.f and c.d.f of a standard normal variate. A distribution with p.d.f (2.1) hereafter we denoted as ACND(λ,α,β).

For some particular choices of α, λ and β, the p.d.f. $f(x;\lambda,\alpha,\beta)$ given in (2.1) of $ACND(\lambda,\alpha,\beta)$ is plotted as given in Figure 1.

Result 2.1. If X has ACND(λ,α,β), then $Y_1 = -X$ has ACND$(-\lambda,\alpha,\beta)$

Proof. The p.d.f $f_1(y)$ of Y_1 is the following, for $y \in R, \lambda \in R, \beta \in R$ and $\alpha \geq -1$.

$$\begin{aligned} f_1(y) &= f(-y;\lambda,\alpha,\beta)\left|\frac{dx}{dy}\right| \\ &= \frac{\phi(-y)}{\alpha+2}\phi(-y)\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(-y)}{\sqrt{1+\lambda^2y^2}}\right)\right] \\ &= f(y;-\lambda,\alpha,\beta) \end{aligned}$$

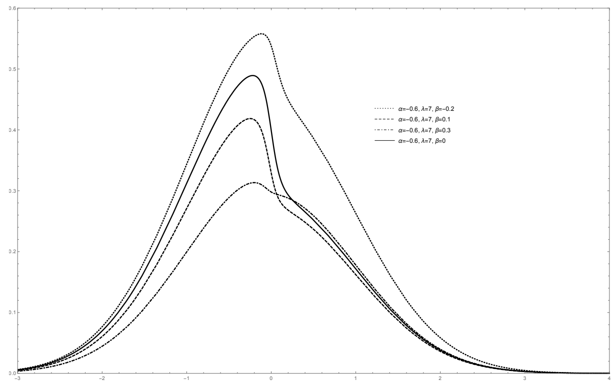

Figure 1: Probability plots of ACND(λ, α, β) for fixed values of α, λ and various values of β

$\square$

Result 2.2. If X has ACND(λ, α, β) then $Y_2 = |X|$ has the p.d.f (2.2), in which

$$\Delta(y) = \Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda y}{\sqrt{1+\lambda^2 y^2}}\right) + \Phi\left(\beta\sqrt{1+\lambda^2} + \frac{-\lambda y}{\sqrt{1+\lambda^2 y^2}}\right).$$

Proof. The p.d.f. $f_2(y)$ of $Y_2 = |X|$ is the following, for $y > 0$.

$$\begin{aligned} f_2(y) &= f(y;\lambda,\alpha,\beta)\Big|\frac{dx}{dy}\Big| + f(-y;\lambda,\alpha,\beta)\Big|\frac{dx}{dy}\Big| \\ &= f(y;\lambda,\alpha,\beta) + f(y;-\lambda,\alpha,\beta) \\ &= \frac{\phi(y)}{\alpha+2}\Big[2 + \alpha[\Phi(\beta)]^{-1}\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda y}{\sqrt{1+\lambda^2 y^2}}\Big)\Big] \\ &\quad + \frac{\phi(y)}{\alpha+2}\Big[2 + \alpha[\Phi(\beta)]^{-1}\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{-\lambda y}{\sqrt{1+\lambda^2 y^2}}\Big)\Big] \\ &= \frac{\phi(y)}{\alpha+2}\Big[4 + \alpha[\Phi(\beta)]^{-1}\Big\{\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda y}{\sqrt{1+\lambda^2 y^2}}\Big) + \Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{-\lambda y}{\sqrt{1+\lambda^2 y^2}}\Big)\Big\}\Big] \\ &= \frac{\phi(y)}{\alpha+2}\big[4 + \alpha[\Phi(\beta)]^{-1}\Delta(y)\big] \qquad (2.2) \end{aligned}$$

$\square$

Result 2.3. If X has ACND(λ, α, β), then $Y_3 = X^2$ has pdf (2.3), in which $\Delta(y)$ is as defined in Result 2.2.

Proof. For $y > 0$, the p.d.f of $g_3(y)$ of Y_3 is

$$\begin{aligned}
f_3(y) &= f(\sqrt{y};\lambda,\alpha,\beta)\Big|\frac{dx}{dy}\Big| + f(-\sqrt{y};\lambda,\alpha,\beta)\Big|\frac{dx}{dy}\Big| \\
&= \frac{\phi(\sqrt{y})}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda\sqrt{y}}{\sqrt{1+\lambda^2 y}}\right)\right]\frac{1}{2\sqrt{y}} \\
&\quad + \frac{\phi(-\sqrt{y})}{\alpha+2}\left[2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{-\lambda\sqrt{y}}{\sqrt{1+\lambda^2 y}}\right)\right]\frac{1}{2\sqrt{y}} \\
&= \frac{\phi(\sqrt{y})}{(\alpha+2)2\sqrt{y}}\left[4+\alpha[\Phi(\beta)]^{-1}\Delta(\sqrt{y})\right] \qquad (2.3)
\end{aligned}$$

□

Result 2.4. The c.d.f of ACND(λ,α,β) with p.d.f (2.1) is the following, for $x \in R$.

$$F(x) = \frac{\Phi(x)}{\alpha+2}\left[2+\alpha\frac{[\Phi(\beta)]^{-1}}{2}\right] - \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta\left(x,\frac{\lambda t}{\sqrt{1+\lambda^2t^2}}\right), \qquad (2.4)$$

where

$$\xi_\beta(x,\lambda) = \int_x^\infty \int_0^{\beta\sqrt{1+\lambda^2}+\frac{\lambda t}{\sqrt{1+\lambda^2 t^2}}} \phi(t)\phi(u)\,du dt$$

can be easily computed using the mathematical softwares such as MATHCAD, MATHEMATICA, etc.

Proof.

$$\begin{aligned}
F(x) &= \int_{-\infty}^x f(t;\lambda,\alpha,\beta)dt \\
&= \frac{2}{\alpha+2}\Phi(x) + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\int_{-\infty}^x \phi(t)\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda t}{\sqrt{1+\lambda^2t^2}}\right)dt \\
&= \frac{2\Phi(x)}{\alpha+2} + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\left[\frac{1}{2}\Phi(x) - \xi_\beta\left(x,\frac{\lambda t}{\sqrt{1+\lambda^2t^2}}\right)\right] \\
&= \frac{\Phi(x)}{\alpha+2}\left[2+\frac{\alpha[\Phi(\beta)]^{-1}}{2}\right] - \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta\left(x,\frac{\lambda t}{\sqrt{1+\lambda^2t^2}}\right)
\end{aligned}$$

□

3 Characteristic Function and Moments

Result 3.1. The characteristic function $\psi_X(t)$ of ACND(λ,α,β) with p.d.f (2.1) is the following, for any $t \in R$ and $i = \sqrt{-1}$.

$$\psi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left\{2+\alpha[\Phi(\beta)]^{-1}E\left[\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(u+it)}{\sqrt{1+\lambda^2(u+it)^2}}\right)\right]\right\}, \qquad (3.1)$$

Proof. Let X follows ACND(λ, α, β) with p.d.f (2.1). Then by the definition of characteristic function we have the following, for any $t \in R$ and $i = \sqrt{-1}$.

$$\begin{aligned}\psi_X(t) &= E(e^{itX}) \\ &= \frac{2}{\alpha+2}\int_{-\infty}^{\infty} e^{itx}\phi(x)dx + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\int_{-\infty}^{\infty} e^{itx}\phi(x)\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}\right)dx \\ &= \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-it)^2}{2}}\Phi(\beta\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2x^2}})dx\right\} \qquad (3.2)\end{aligned}$$

On substituting $x - it = u$, in (3.2) we obtain

$$\psi_X(t) = \frac{e^{-t^2/2}}{\alpha+2}\left\{2 + \alpha[\Phi(\beta)]^{-1}E\left[\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda(u+it)}{\sqrt{1+\lambda^2(u+it)^2}}\right)\right]\right\}, \qquad (3.3)$$

which implies (3.1). □

The expression for even moments and odd moments of $ACND(\lambda, \alpha, \beta)$ is given in the following results.

Result 3.2. If X follows $ACND(\lambda, \alpha, \beta)$ then for any $k = 1, 2, \ldots$

$$E(X^{2k}) = \frac{2^{k+\frac{1}{2}}}{(\alpha+2)\sqrt{2\pi}}\Gamma(k+\frac{1}{2}) + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}A_k(\beta,\lambda), \qquad (3.4)$$

in which

$$A_k = \int_0^{\infty} u^{k-\frac{1}{2}}\phi(u)\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{1+\lambda^2 u}}\right)du,$$

for $\lambda \in R$, $\beta \in R$ which can be easily evaluated by using the softwares such as MATHCAD, MATHEMATICA, etc.

Proof. By the definition of raw moments

$$E(X^{2k}) = \int_{-\infty}^{\infty} x^{2k} f(x; \lambda, \alpha, \beta)dx. \qquad (3.5)$$

On substituting $x^2 = u$ in (3.5) to obtain,

$$\begin{aligned}E(X^{2k}) &= \int_0^{\infty} u^k\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}\int_0^{\infty} u^k\phi(\sqrt{u})\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{(1+\lambda^2 u)}}\Big)\frac{1}{\sqrt{u}}du \\ &= \frac{1}{(\alpha+2)}\int_0^{\infty}\Big[u^{k-\frac{1}{2}}f(\sqrt{u})du + \frac{\alpha[\Phi(\beta)]^{-1}}{2}u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{1+\lambda^2 u}}\Big)\Big]du,\end{aligned}$$

which leads to (3.4). □

Result 3.3. If X follows $ACND(\lambda, \alpha, \beta)$ then for any k=0,1,2,...

$$E(X^{2k+1}) = \frac{2^{k+1}}{(\alpha+2)\sqrt{2\pi}}\Gamma(k+1) + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}B_k(\beta,\lambda) \tag{3.6}$$

in which

$$B_k = \int_0^\infty u^k \phi(u)\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{1+\lambda^2 u}}\right)du,$$

for $\lambda \in R$, $\beta \in R$, which can be easily evaluated by using the softwares such as MATHCAD, MATHEMATICA, etc.

Proof. By the definition of raw moments

$$E(X^{2k+1}) = \int_{-\infty}^{\infty} x^{2k+1} f(x;\alpha,\beta,\lambda)dx. \tag{3.7}$$

On substituting $x^2 = u$ in (3.5) we get,

$$\begin{aligned} E(X^{2k+1}) &= \int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\frac{1}{\sqrt{u}}\Big[1 + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{(1+\lambda^2 u)}}\Big)\Big]du \\ &= \frac{1}{(\alpha+2)}\left[\int_0^\infty u^k f(\sqrt{u})du + \frac{\alpha[\Phi(\beta)]^{-1}}{2(\alpha+2)}u^k\phi(\sqrt{u})\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda\sqrt{u}}{\sqrt{1+\lambda^2 u}}\right)\right]du, \end{aligned}$$

which implies (3.6). □

4 Reliability Measures and Mode

In this section we obtain some properties of ACND(λ, α, β) with p.d.f. (2.1) useful in reliability studies. Let X follows ACND(λ, α, β) with p.d.f (2.1). Now, from the definition of reliability function $R(t)$, failure rate $r(t)$ and mean residual life function $\mu(t)$ of X, we obtain the following results.

Result 4.1. The reliability function $R(t)$ of X is the following, in which $\xi_\beta(t, \frac{\lambda x}{\sqrt{1+\lambda^2x^2}})$ is as defined in Result 2.4.

$$R(t) = \frac{[1-\Phi(t)]}{\alpha+2}\left\{2 + \frac{\alpha[\Phi(\beta)]^{-1}}{2}\right\} + \frac{\alpha[\Phi(\beta)]^{-1}}{\alpha+2}\xi_\beta\left(t, \frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right)$$

Result 4.2. The failure rate $r(t)$ of X is given by

$$r(t) = \frac{\phi(t)[2 + \alpha[\Phi(\beta)]^{-1}\Phi(\beta\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2x^2}})]}{(1-\Phi(t))[2 + \frac{\alpha[\Phi(\beta)]^{-1}}{2}] + \alpha[\Phi(\beta)]^{-1}\xi_\beta(t, \frac{\lambda x}{\sqrt{1+\lambda^2x^2}})}.$$

Result 4.3. The mean residual life function of ACND(λ, α, β) is

$$M(t) = \frac{2\phi(t)}{(\alpha+2)R(t)} + \frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)}\Big[\Phi\Big(\beta\sqrt{1+\lambda^2} + \frac{\lambda t}{\sqrt{1+\lambda^2t^2}}\Big)\phi(t) + \xi^*_\beta(t;\lambda)\Big] - t, \tag{4.1}$$

where

$$\xi^*_\beta(t;\lambda) = \int_t^\infty \phi(x) \left[\frac{d}{dx} \left(\int_0^{\beta\sqrt{1+\lambda^2}+\frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}} \phi(u)du \right) \right] dx$$

Proof. By definition, the mean residual life function (MRLF) of X is given by

$$M(t) = E(X - t \,|\, X > t) = E(X \,|\, X > t) - t, \tag{4.2}$$

where

$$\begin{aligned} E(X \,|\, X > t) &= \frac{2}{R(t)(\alpha+2)} \int_t^\infty x\phi(x)dx + \frac{\alpha[\Phi(\beta)]^{-1}}{R(t)} \int_t^\infty x\phi(x)\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right) dx \\ &= \frac{2}{(\alpha+2)R(t)} \int_t^\infty -\phi^{'}(x)dx + \frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)} \int_t^\infty -\phi^{'}(x)\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right) dx \\ &= \frac{2}{(\alpha+2)R(t)}\phi(t) + \frac{\alpha[\Phi(\beta)]^{-1}}{(\alpha+2)R(t)} \left(-\Phi(\lambda x + \beta\sqrt{1+\lambda^2})\phi(x)\right)_t^\infty \\ &\quad - \frac{\alpha[\Phi(\beta)]^{-1}}{R(t)(\alpha+2)} \int_t^\infty -\phi(x) \left[\frac{d}{dx} \left(\int_{-\infty}^{\beta\sqrt{1+\lambda^2}+\frac{\lambda x}{\sqrt{1+\lambda^2x^2}}} \phi(u)du \right) \right] dx \end{aligned} \tag{4.3}$$

On solving (4.3) and substituting in (4.2), we get (4.1). The functions $R(t)$, $r(t)$ and $M(t)$ are equivalent in the sense that if one of them is given the other two can be uniquely determined. □

Result 4.4. Case 1: For $x > 0$, the p.d.f of $ACND(\lambda, \alpha, \beta)$ is log concave

(i) if $\lambda < 0$, provided for all $\alpha > 0$ and $\beta > 0$ and

(ii) if $\lambda > 0$, provided $\left|\frac{3\lambda^5x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}\right| < \left|\frac{3\lambda^3x}{(1+\lambda^2x^2)^{\frac{3}{2}}}\right|$

Case 2: For $x < 0$, the p.d.f of $ACND(\lambda, \alpha, \beta)$ is log concave

(i) if $\lambda > 0$, provided for all $\alpha > 0$ and $\beta > 0$ and

(i) if $\lambda < 0$, provided $\left|\frac{3\lambda^5x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}\right| > \left|\frac{3\lambda^3x}{(1+\lambda^2x^2)^{\frac{3}{2}}}\right|$.

To establish $\log[f(x;\lambda,\alpha,\beta)]$ is a concave function of x, it is enough to show that its second derivative is negative for all x. Thus,

$$\begin{aligned} &\frac{d}{dx}\log[f(x;\lambda,\alpha,\beta)] = -x + \frac{\alpha[F(\beta)]^{-1}f(h)h^{'}}{2+\alpha[F(\beta)]^{-1}F(h)} \quad \text{and} \\ &\frac{d^2}{dx^2}\log[f(x;\lambda,\alpha,\beta)] = -1 - B_1 - B_2 + B_3 \end{aligned}$$

in which

$$B_1 = \frac{\alpha[F(\beta)]^{-1}h^{'2}f(h)h}{2+\alpha[F(\beta)]^{-1}F(h)}, \quad B_2 = \frac{\alpha^2[F(\beta)]^{-2}(f(h))^2h^{'2}}{[2+\alpha[F(\beta)]^{-1}F(h)]^2}, \quad B_3 = \frac{\alpha[F(\beta)]^{-1}f(h)h^{''}}{2+\alpha[F(\beta)]^{-1}F(h)}$$

where

$$h = \frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}} + \beta\sqrt{1+\lambda_1^2},\ h^{'} = \frac{\lambda_1}{\sqrt{1+\lambda_2 x^2}} - \frac{\lambda_1\lambda_2 x^2}{(1+\lambda_2 x^2)^{\frac{3}{2}}},\ h^{''} = \frac{3\lambda^5 x^3}{(1+\lambda^2 x^2)^{\frac{5}{2}}} - \frac{3\lambda^3 x}{(1+\lambda^2 x^2)^{\frac{3}{2}}}$$

Note that $B_1 > 0$ for $\alpha > 0$ and $h > 0$. And $h > 0$ for all values of $\lambda, \beta > 0$. Consequently $B_2 > 0$ for all values of $\lambda, \alpha, \beta > 0$. Also $B_3 < 0$ for either $\alpha < 0$ and $h^{''} > 0$ or $\alpha > 0$ and $h^{''} < 0$. Hence (2.1) is log concave in these situations.

Result 4.5. ACND(λ, α, β) density is strongly unimodal under the following two cases.
Case 1: For $x > 0$,

(i) if $\lambda < 0$, provided for all $\alpha > 0$ and $\beta > 0$ and

(ii) if $\lambda > 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2 x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2 x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda > 0$, provided for all $\alpha > 0$ and $\beta > 0$ and

(i) if $\lambda < 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2 x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2 x^2)^{\frac{3}{2}}}|$.

Result 4.6. ACND(λ, α, β) density is plurimodal under the following two cases.
Case 1: For $x > 0$,

(i) if $\lambda < 0$, provided for all $\alpha < 0$ and $\beta > 0$ and

(ii) if $\lambda > 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2 x^2)^{\frac{5}{2}}}| > |\frac{3\lambda^3 x}{(1+\lambda^2 x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda > 0$, provided for all $\alpha < 0$ and $\beta > 0$ and

(i) if $\lambda < 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2 x^2)^{\frac{5}{2}}}| > |\frac{3\lambda\lambda^2 x}{(1+\lambda^2 x^2)^{\frac{3}{2}}}|$.

5 Location Scale Extension

In this section we discuss an extended form of ACND(λ, α, β) by introducing the location parameter μ and scale parameter σ.

Definition 5.1. Let $X \sim ACND(\lambda, \alpha, \beta)$ with p.d.f given in (2.1). Then $Y = \mu + \sigma X$ is said to have an extended ACND with the following p.d.f.

$$f^*(y;\mu,\sigma,\lambda,\alpha,\beta) = \frac{\phi\left(\frac{y-\mu}{\sigma}\right)}{\sigma(\alpha+2)}\Big[2+\alpha[\Phi(\beta)]^{-1}\Phi\Big(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}}\Big)\Big], \quad (5.1)$$

in which $y \in R$, $\mu \in R$, $\lambda \in R$, $\beta \in R$, $\sigma > 0$, $\lambda \geq 0$ and $\alpha \geq -1$. A distribution with p.d.f (5.1) is denoted as EACND$(\mu, \sigma; \lambda, \alpha, \beta)$. Clearly when

(i) $\beta = 0$ and $\lambda = 0$ EACND$(\mu, \sigma; \lambda, \alpha, \beta)$ reduces to the p.d.f of normal distribution.

Now, we obtain the following results of EACND$(\mu, \sigma; \lambda, \alpha, \beta)$, in a similar way as we defined in section 2, 3 and 4.

Result 5.1. The c.d.f $F^*(y)$ of EACND$(\mu, \sigma; \lambda, \alpha, \beta)$ with p.d.f (5.1) is the following, for $y \in R$.

$$F^*(y) = \left[2 + \frac{\alpha\Phi[(\beta)]^{-1}}{2}\right] \frac{\Phi(\frac{y-\mu}{\sigma})}{\sigma(\alpha+2)} - \frac{\alpha[\Phi(\beta)]^{-1}}{\sigma(\alpha+2)} \xi^*_\beta \left(y, \frac{\lambda(t-\mu)}{\sqrt{\sigma^2 + \lambda^2(t-\mu)^2}}\right)$$

where $\xi^*_\beta \left(y, \frac{\lambda(t-\mu)}{\sqrt{\sigma^2+\lambda^2(t-\mu)^2}}\right)$ is as defined in Result 2.4.

Result 5.2. The characteristic function of EACND$(\mu, \sigma; \lambda, \alpha, \beta)$ is given by

$$\psi^*_Y(t) = \frac{e^{it\mu - \frac{t^2\sigma^2}{2}}}{\alpha+2} \left\{2 + \alpha[\Phi(\beta)]^{-1} E\left[\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda(z+\sigma^2 it)}{\sqrt{\sigma^2 + \lambda^2(z+\sigma^2 it)^2}}\right)\right]\right\}.$$

Result 5.3. The reliability function $R^*(t)$ of Y is the following, in which $\xi^*_\beta(t, \frac{\lambda(y-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}})$ is as defined in Result 2.4.

$$R^*(t) = \frac{1}{\sigma(\alpha+2)}\left[1 - F(\frac{t-\mu}{\sigma})\right]\left\{2 + \frac{\alpha}{2}[F(\beta)]^{-1}\right\} + \frac{\alpha[F(\beta)]^{-1}}{\sigma(\alpha+2)}$$
$$\xi^*_\beta\left(t, \frac{\lambda(y-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}}\right)$$

Result 5.4. The failure rate $r^*(t)$ of Y is given by

$$r^*(t) = \frac{f(\frac{t-\mu}{\sigma})\left[2 + \alpha[F(\beta)]^{-1}F\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda_1(t-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}}\right)\right]}{\frac{1}{\sigma(\alpha+2)}\left[1 - F(\frac{t-\mu}{\sigma})\right]\left\{2 + \frac{\alpha}{2}[F(\beta)]^{-1}\right\} + \frac{\alpha[F(\beta)]^{-1}}{\sigma(\alpha+2)}\xi^*_\beta\left(t, \frac{\lambda(y-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}}\right)}$$

6 Maximum Likelihood Estimation

The log likelihood function, ln L of the random sample of size n from a population following EMACND$(\mu, \sigma; \lambda, \alpha, \beta)$ is the following,

$$ln\, L = n\log\left(\frac{1}{\sqrt{2\pi}}\right) - n\log\sigma - n\log(\alpha+2) - \frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^2}$$
$$+ \sum_{i=1}^{n}\log\left(2 + \alpha\left[\Phi(\beta)\right]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2} + \frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\right) \quad (6.1)$$

On differentiating (6.1) with respect to parameters $\mu, \sigma, \beta, \lambda$ and α and then equating to zero, we obtain the following normal equations.

$$\sum_{i=1}^{n} \frac{(y_i-\mu)}{\sigma^2} - \sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\left(\frac{\lambda}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)}$$
$$+\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\left(\frac{\lambda^3(y_i-\mu)^2}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)} = 0, \tag{6.2}$$

$$\frac{n}{\sigma} - \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{\sigma^3}$$
$$-\sum_{i=1}^{n} \frac{\alpha\lambda\Phi[(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\left(\frac{(y_i-\mu)\sigma}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)} = 0, \tag{6.3}$$

$$\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\left[\frac{y_i-\mu}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}+\frac{\beta\lambda}{\sqrt{1+\lambda^2}}\right]}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)} = 0, \tag{6.4}$$

$$\sum_{i=1}^{n} \frac{\alpha[\Phi(\beta)]^{-1}\phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)\left[\frac{\lambda(y_i-\mu)^3}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}}\right]}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)}$$
$$\left(\frac{\beta\lambda}{\sqrt{1+\lambda^2}} - \frac{\lambda^2(y_i-\mu)^3}{(\lambda^2(y_i-\mu)^2+\sigma^2)^{\frac{3}{2}}} + \frac{y_i-\mu}{\sqrt{\lambda^2(y_i-\mu)^2+\sigma^2}}\right) = 0, \tag{6.5}$$

and

$$\sum_{i=1}^{n} \frac{[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)}{2+\alpha[\Phi(\beta)]^{-1}\Phi\left(\beta\sqrt{1+\lambda^2}+\frac{\lambda(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)} = 0. \tag{6.6}$$

On solving the equations (6.2) to (6.6), we get the maximum likelihood estimate of the parameters of EACND($\mu, \sigma; \lambda, \alpha, \beta$).

7 Generalized Likelihood Ratio Test

In this section we discuss a test procedure for testing the parameter β of $EACND$. For testing the null hypothesis $H_0 : \beta = 0$ against the alternative hypothesis $H_1 : \beta \neq 0$ by using the generalized likelihood ratio test, the test statistic is

$$-2ln\lambda(x) = 2[lnL(\hat{\Theta}; x) - lnL(\hat{\Theta}^*; x)], \tag{7.1}$$

where $\hat{\Theta}$ is the maximum likelihood estimator of $\Theta = (\mu, \sigma, \lambda, \alpha, \beta)$ with no restriction, and $\hat{\Theta}^*$ is the maximum likelihood estimator of Θ when $\beta = 0$. The test statistic given is asymptotically distributed as χ^2 with 1 degrees of freedom.

8 Applications

In this section we consider two real life data application of EACND. The first data concerning the heights (in centimeters) of 100 Australian athletes, given in Cook and Weisberg (1994). The second data represent the lean body mass of Australian athletes. The data given in Cook and Weisberg (1994). We obtained the maximum likelihood estimate (MLE) of the parameters by using these data sets with the help of the MATHCAD software. The numerical results obtained are presented in Table 1, which includes the estimated values of the parameters and the corresponding Kolmogorov Smirnov Statistics (KSS) values of models ESCND($\mu, \sigma; \lambda, \alpha$) and EACND($\mu, \sigma; \lambda, \alpha, \beta$). Also its AIC, BIC and $AICc$ values are obtained and included in Table 1.

Table 1: Estimated values of the parameters for the model: ESCND($\mu, \sigma; \lambda, \alpha$) and EACND($\mu, \sigma;$ λ, α, β) with respective values of KSS, AIC, BIC and AICc in case of Data sets 1 and 2.

Data	method	μ	σ	λ	β	α	KSS	P-value	AIC	BIC	AICc
1	ESCND	172.01	3.72	1.84	-	4.12	0.6	$<$ .0001	830.96	841.38	831.38
	EACND	174.59	8.23	0.81	10	2	0.09	0.37	714.39	727.42	715.03
2	ESCND	54.21	4.41	12.94	-	0.45	0.28	$<$ .0001	685.74	696.16	686.16
	EACND	54.90	6.92	10	8.4	0.78	0.11	0.18	679.73	692.76	680.37

It is clear from Table 1 that the EACND($\mu, \sigma; \lambda, \alpha, \beta$) is a more appropriate model to both the data sets compared to the existing model ESCND($\mu, \sigma; \lambda, \alpha$). Also, we have plotted the histogram of Data sets 1 and 2 along with the fitted probability plots corresponding to the $EACND$ and $ESCND$ in Figures 2 and 3. From the figures it can be seen that the $EACND$ yields a better fit compared to the $ESCND$ in case of both the Data sets 1 and 2. Thus, the model discussed in this paper provides more flexibility in modeling. Also we conduct a generalized likelihood ratio test for illustrating the suitability of the model EACND, which is described as follows.

Let us consider the problem of testing the hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ in the case of Data set 1. The computed values of the MLEs and likelihood of the distributions $ESCND$ and

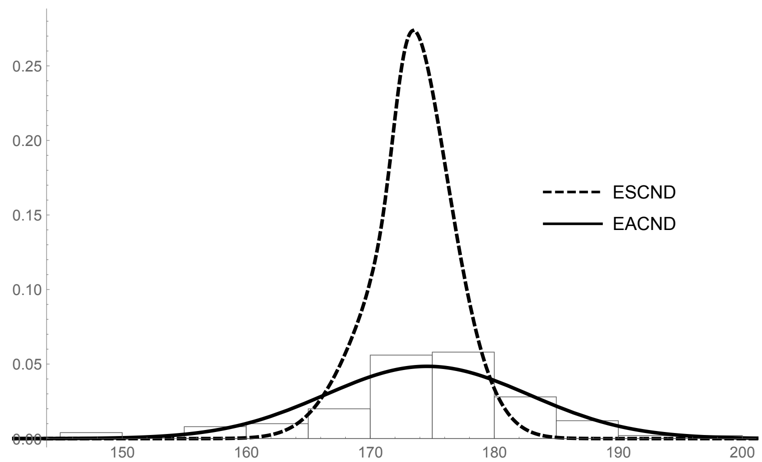

Figure 2: Histogram of Data set 1 and fitted distributions

$EACND$ are as follows.

$$\hat{\mu} = 172.005,\ \hat{\sigma} = 3.723,\ \hat{\lambda} = 1.839,\ \hat{\alpha} = 4.118,$$

$L(\hat{\Theta}^*; x) = 1.98091 \times 10^{-179}$ and

$$\hat{\mu} = 174.59,\ \hat{\sigma} = 8.23,\ \hat{\lambda} = 0.809,\ \hat{\beta} = 10,\ \hat{\alpha} = 2,$$

$L(\hat{\Theta}; x) = 1.10608 \times 10^{-153}$. The calculated value of likelihood ratio (LR) test statistic is 118.569. Since the critical value for the test with significance level 0.05 at one degrees of freedom is 3.84, the null hypothesis is rejected.

Similarly we consider the problem of testing $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ using the Data set 2. The MLEs and values of the likelihood of the distributions $ESCND$ and $EACND$ are as follows.

$$\hat{\mu} = 54.209,\ \hat{\sigma} = 4.406,\ \hat{\lambda} = 12.937,\ \hat{\alpha} = 0.451,$$

$L(\hat{\Theta}^*; x) = 6.78635 \times 10^{-148}$ and

$$\hat{\mu} = 54.895,\ \hat{\sigma} = 6.922,\ \hat{\lambda} = 10,\ \hat{\beta} = 8.4,\ \hat{\alpha} = 0.78,$$

$L(\hat{\Theta}; x) = 3.70745 \times 10^{-146}$. The calculated value of likelihood ratio (LR) test statistic is 8.0012. Since the critical value for the test with significance level 0.05 at one degrees of freedom is 3.84, the null hypothesis is rejected.

9 Simulation Study

In order to assess the performance of the maximum likelihood estimators of the parameters of the EACND($\mu, \sigma; \lambda, \alpha, \beta$), we have conducted a brief simulation study by generating observations with

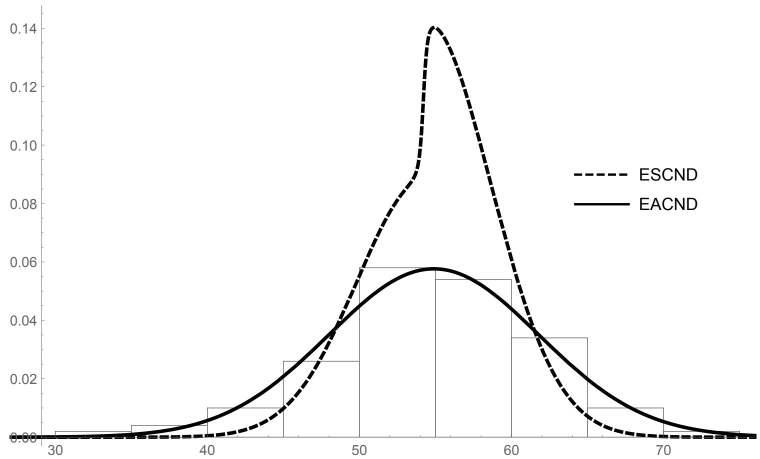

Figure 3: Histogram of Data set 2 and fitted distributions

the help of MATHEMATICA for the following set of parameters $\mu = 2, \sigma = 0.5, \lambda = 0.8, \beta = 6$ and $\alpha = 0.3$. We have considered 200 bootstrap samples of sizes 10, 30, 50 and 70 from the EACND for comparing the performances of the maximum likelihood estimators. The likelihood estimates of the parameters, the average bias estimates and average MSEs over 200 replications are calculated and presented in Table 2.

Table 2: Estimate of the parameters and corresponding bias and mean square error (MSE)

Sample size	Statistics	μ	σ	λ	β	α
10	estimate	2.146392	0.9304701	0.9	6.4 6	0.69
	bias	0.1463921	0.4304701	0.1	0.4	0.39
	MSE	0.02143064	0.1853045	0.01	0.1	0.1521
30	estimate	1.989697	0.4912964	0.89	6.2	0.6
	bias	-0.01030252	-0.008703591	0.09	0.2	0.3
	MSE	0.0001061419	7.57525E-05	0.0081	0.04	0.09
50	estimate	2.004876	0.4924149	0.8	6	0.5
	bias	0.004876226	-0.007585067	-0.01	-0.1	0.2
	MSE	2.377758E-05	5.753324E-05	1E-04	0.01	0.04
70	estimate	2.003085	0.4968208	0.8	6	0.49
	bias	0.003085262	-0.003179247	6.065148E-13	-1.311755E-09	0.19
	MSE	9.518841E-06	1.010761E-05	3.678602E-25	1.720702E-18	0.0361

From Table 2 it can be observed that both the bias and MSE are in decreasing order as sample size increases.

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Received: August 12, 2018

Accepted: February 8, 2019