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TWO-DIMENSIONAL STOCHASTIC MODELING FOR PREDICTING BANKRUPTCY FOR MANUFACTURING COMPANIES

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ABSTRACT

Increasing accuracy of the model prediction on business bankruptcy helps reduce substantial losses for owners, creditors, investors and workers, and, further, minimize an economic and social problem frequently. In this study, we propose a stochastic model of financial working capital and cashflow as a two-dimensional Brownian motion $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ on the business bankruptcy prediction. The probability of bankruptcy occurring in a time interval $[0, T]$ is defined by the boundary crossing probability of the two-dimensional Brownian motion entering a predetermined threshold domain. Mathematically, we extend the result in Fu and Wu (2016) on the boundary crossing probability of a high dimensional Brownian motion to an unbounded convex hull. The proposed model is applied to a real data set of companies in US and the numerical results show the proposed method performs well.

Keywords and phrases: Brownian motion; Boundary crossing probability; Cashflow; Finite Markov chain imbedding; Probability of bankruptcy; Working capital

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1 Introduction

Bankruptcy for a company is an event which will produce substantial losses for owners, creditors, and workers. It may start a chain reaction and can cause serious national and international social problems and economic crises (see Deakin (1972)), for example, the collapse of Lehman Brothers in 2008. Predicting bankruptcy is a serious challenge to both financial institutions and governments. Several statistical methods have been developed to predict business bankruptcy for decades. Beaver (1966) applied a univariate discriminant analysis using financial ratios from accounting data to predict business bankruptcy. The predictive ability of thirty individual financial ratios was studied. Among these thirty ratios, the predictive ability of the ratio of cashflow to total assets was ranked the highest. Altman (1968) extended Beawer's univariate analysis to a multivariate linear discriminant analysis with five ratio variables: (i) working capital to total assets, (ii) sales to total assets, (iii) retained earnings to total assets, (iv) earning before income tax to total assets, and (v) market value of equity to total assets. Deakin (1972) and Blum (1974) utilized the Altman's multivariate linear discriminant analysis approach and developed a business bankruptcy prediction model using different sets of ratios and adding other variables, such as (i) cashflow to total liabilities and (ii) net worth at book value to total liabilities. They claimed that their proposed model provides better predictive accuracy than those proposed by Beaver (1966) and Altman (1968), and further pointed out that the predictive accuracy depends highly on the proper predictor variables. Many also pointed out that the two variables with the highest predictive ability among all the variables they considered are working capital and cashflow. Some studied the bankruptcy by fitting a logistic regression model and claimed that the logistic regression model performs at least as well as the discriminant analysis and proved that working capital a better predictor (Ohlson (1980), Lo (1986), Theodossiou (1993) and Kahya (1997)).

Fu (2009) defined a probability of a company's bankruptcy during the time period $[0, T]$ in terms of a boundary crossing probability (BCP) of a univariate (working capital) Markov chain entering a predetermined domain determined by a threshold. Monte Carlo method was used to find the boundary crossing probability for predicting business bankruptcy. Numerical results showed the method out performed both the multi-discriminant analysis and logistic regression approaches considerably.

In this manuscript we propose a stochastic model based on two most predictive variables, working capital and cashflow $\boldsymbol{X}(t) = (X_1(t), X_2(t))$, as a two-dimensional Brownian motion with a linear drift $t\boldsymbol{\beta}$ and covariance matrix $t\boldsymbol{\Sigma}$ for predicting the probability of bankruptcy as a probability of the process entering a predetermined threshold domain. Mathematically, we extend the recent result Fu and Wu (2016) on the boundary crossing probability of a high dimensional Brownian motion from a compact to an unbounded convex hull by using the finite Markov chain imbedding (FMCI) technique. It provides a simple and efficient way to evaluate the probability of bankruptcy. The mathematical detail for finding the probability of bankruptcy is given in Section 2. A numerical example for predicting the bankruptcy of manufacturing companies is given in Section 3. Discussions, conclusions and possible extensions are given in Section 4.

2 Probability of Bankruptcy

Throughout this manuscript we assume that the stochastic process $\boldsymbol{X}(t) = (X_1(t),\ X_2(t))$ of working capital and cashflow be a two-dimensional and correlated Brownian motion which starts at $\boldsymbol{X}(0) = (x_1, x_2)$ with a linear trend (drift) $t\boldsymbol{\beta} = (t\beta_1, t\beta_2)$ and has a covariance matrix $t\boldsymbol{\Sigma}$ over the time interval $[0, T]$. Given two thresholds a and c (constant boundaries), we define an unbounded convex hull

$$\boldsymbol{B}(t) = \{(x_1, x_2) \in R^2 : x_1 \geq a \text{ and } x_2 \geq c,\ \forall t \in [0, T]\}$$

of $\mathbb{R}^2 \times [0, T]$ and the probability of bankruptcy is determined by

$$\gamma(\overline{\text{int}\boldsymbol{B}(t)}) = 1 - P\big(\boldsymbol{X}(t) \in \text{int}\boldsymbol{B}(t),\ \forall\ t \in [0, T]\,|\,\boldsymbol{X}(0), \boldsymbol{\beta},\ \text{and}\ \boldsymbol{\Sigma}\big)$$

as the boundary crossing probability. Note that $\boldsymbol{B}(t)$ is an unbounded convex hull. To the best of our knowledge, there are no general results on BCP of a high dimensional Brownian motion for an unbounded convex hull.

For the convenience of developing the mathematical results, we study the probability of the process $\boldsymbol{X}(t)$ staying in the region $\text{int}\boldsymbol{B}(t)$ without drifting into the threshold domain $\overline{\text{int}\boldsymbol{B}(t)}$ during the time interval $[0, T]$, i.e.,

$$P\left(\boldsymbol{X}(t) \in \text{int}\boldsymbol{B}(t),\ \forall t \in [0, T]\,|\,\boldsymbol{X}(0), \boldsymbol{\beta},\ \text{and}\ \boldsymbol{\Sigma}\right).$$

Let $\boldsymbol{W}(t) = (W_1(t), W_2(t)), t \geq 0$ be a standard two-dimensional Brownian motion with drift $\boldsymbol{\beta} = \boldsymbol{0}$ and covariance matrix $t\boldsymbol{\Sigma} = t\boldsymbol{I}$. Throughout the manuscript, we assume $P(\boldsymbol{W}(0) = (0, 0)) = 1$ and let $A(t) \subseteq \mathbb{R}^2 \times [0, T]$ be a compact convex hull which satisfies the following conditions:

(A) $A(t)$ is a compact convex hull with respect to product topology in $\mathbb{R}^2 \times [0, T]$ (see Fu and Wu (2016)),

(B) $\partial A(t)$, the boundary of $A(t)$, is a continuous function in $t \in [0, T]$ and satisfies Lipschitz conditions, and

(C) $(0, 0) \in \text{int}A(0)$.

Recently Fu and Wu (2016) constructed a discrete two-dimensional Markov chain $\hat{\boldsymbol{W}}_n(t)$ which converges to a two dimensional standard Brownian motion in distribution; i.e.,

$$\hat{\boldsymbol{W}}_n(t) \xrightarrow{\mathcal{D}} \boldsymbol{W}(t) \ \text{ as } n \to \infty,$$

for all $t \in [0, T]$ and showed that the passing through probability

$$P(\boldsymbol{W}(t) \in \text{int}A(t), \ \text{ for all } \ t \in [0, T])$$

is finite Markov chain imbeddable in the following sense.

Theorem 1. *If $A(t)$ satisfies the conditions (A), (B) and (C), then there exists a Markov chain $\{Y_i\}_{i=1}^n$ induced by the Markov chain $\hat{\boldsymbol{W}}_n(t)$ such that*

$$P\left(\boldsymbol{W}(t) \in intA(t),\ \forall t \in [0,T]\right) = \lim_{n\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^n \boldsymbol{N}_i\right) \boldsymbol{1}',$$

where $\boldsymbol{\xi}_0$ is the initial distribution of Y_0, $\boldsymbol{N}_i$, $i = 1, \ldots, n$ are the essential transition probability matrices of Markov chain $\{Y_i\}_{i=1}^n$ and $\boldsymbol{1} = (1, \cdots, 1)$, and the rate of convergence is

$$\left| P\left(\boldsymbol{W}(t) \in intA(t),\ \forall t \in [0,T]\right) - \boldsymbol{\xi}_0 \left(\prod_{i=1}^n \boldsymbol{N}_i\right) \boldsymbol{1}' \right| = O\left(\frac{1}{\sqrt{n}}\right). \tag{2.1}$$

We shall not repeat the detailed construction of $\hat{\boldsymbol{W}}_n(t), N_i$ and the proof of Theorem 1 here (see Fu and Wu (2016) for the details of the proof).

It is well-known that the linear transformation

$$\varphi : \boldsymbol{W}(t) = \left(\boldsymbol{X}(t) - \boldsymbol{X}(0) - t\boldsymbol{\beta}\right) \boldsymbol{\Sigma}^{-1/2}$$

yields a standard Brownian motion. It follows from Walsh (1921) that the set

$$\begin{aligned} D(t) &= \varphi(\boldsymbol{B}(t)) \\ &= \left\{\boldsymbol{W}(t) : \boldsymbol{w}(t) = \left(\boldsymbol{x}(t) - \boldsymbol{x}(0) - t\boldsymbol{\beta}\right) \boldsymbol{\Sigma}^{-1/2} : \boldsymbol{x}(t) \in \boldsymbol{B}(t),\ \forall t \in [0,T]\right\} \end{aligned}$$

is also an unbounded convex hull in $\mathbb{R}^2 \times [0,T]$. Given a large l, we define a compact convex hull

$$S(t,l) = \{\boldsymbol{W}(t) : -l \le W_1(t) \le l, -l \le W_2(t) \le l, \forall t \in [0,T]\}.$$

Let

$$A(t,l) = S(t,l) \cap D(t).$$

Note that the set $A(t,l)$ is also a compact convex hull in $\mathbb{R}^2 \times [0,T]$.

Lemma 2.1. *When $l \to \infty$, we have*

$$\begin{aligned} &\left|P\left(\boldsymbol{W}(t) \in intD(t),\ \forall t \in [0,T]\right) - P\left(\boldsymbol{W}(t) \in intA(t,l),\ \forall t \in [0,T]\right)\right| \\ &\qquad = o(\exp\{-l\}). \end{aligned} \tag{2.2}$$

Proof. For one dimensional case, it is well known (see Taylor and Karlin (1998)) that

$$\begin{aligned} P\left(W_1(t) > l,\ \text{for some } t \in [0,T]\right) &= \frac{1}{\sqrt{2\pi T}} \int_l^\infty \exp\left\{-x^2/2T\right\} dx \\ &= o(\exp\{-l\}) \end{aligned} \tag{2.3}$$

as $l \to \infty$. The above result (2.2) follows immediately from the inequality

$$\begin{aligned}&|P\left(\boldsymbol{W}(t) \in \text{int}D(t),\ \forall t \in [0,T]\right) - P\left(\boldsymbol{W}(t) \in \text{int}A(t,l),\ \forall t \in [0,T]\right)| \\ &\leq 4P\left(W_1(t) > l,\ \text{for some } t \in [0,T]\right)\end{aligned}$$

and the equation (2.3). This completes the proof. □

Theorem 2. *The probability of bankruptcy*

$$\gamma(\overline{int\boldsymbol{B}(t)}) = 1 - \lim_{l\to\infty}\lim_{n\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i\right) \mathbf{1}'. \tag{2.4}$$

Proof. It follows from Lemma 2.1 that

$$\left|\gamma(\overline{\text{int}\boldsymbol{B}(t)}) - \gamma(\overline{\text{int}A(t,l)})\right| \to 0 \tag{2.5}$$

as $l \to \infty$. Given l, it follows from Theorem 1 that

$$\gamma(\overline{\text{int}D(t)}) = \gamma(\overline{\text{int}A(t,l)}) = 1 - \lim_{n\to\infty} \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i\right) \mathbf{1}'. \tag{2.6}$$

The result in (2.6) is a direct consequence of Theorem 1 and Equation (2.5). Taking large l, Eq. (2.4) follows immediately from the triangle inequality, Lemma 2.1 and Theorem 1. □

The above result implies that, for large l and n, the probability of bankruptcy $\gamma(\overline{\text{int}\boldsymbol{B}(t)})$ can be approximated by $1 - \boldsymbol{\xi}_0 \left(\prod_{i=1}^{n} \boldsymbol{N}_i\right) \mathbf{1}'$.

Remark 1. Note that the Theorem 2 shows that the BCP for a two-dimensional unbounded convex hull can be approximated by the BCP for a compact convex hull. This result can be easily extended to a higher dimensional ($d > 2$) unbounded convex hull. This is an important character of the FMCI approach for approximating BCP.

3 Predicting Bankruptcy

We consider 178 small manufacturing companies operating in year 2014 and having revenue less than or equal to 25 millions from north American companies in the Compustat Merged Database. If the financial data of a company was available in 2015, the company is deemed operational in 2015. Similarly, if data was not available, the company is deemed bankrupt. At least 12 quarterly of working capital and cash flow collected before 2014 were used for estimating the linear trend $t\boldsymbol{B}$ and covariance matrix $\boldsymbol{\Sigma}$ by using the ordinary least squares method. For some given thresholds in working capital and cashflow, a company is predicted bankrupt if its BCP is higher than or equal to 0.8 and operational otherwise. Numerical results based on two sets of thresholds (-2.0TA, -1.5TA) and (-2.5TA, -2.0TA) are provided in Tables 1 and 2. Note that TA means total amount of the assets of the company.

Table 1: Prediction performance of 178 companies given thresholds $(-2.0TA, -1.5TA)$

Company Status in 2015	Prediction Status in 2015 Operating	Bankrupt	Total
Operating	161	3	164
Bankrupt	6	8	14
Total	167	11	178

Table 2: Prediction performance of 178 companies given thresholds $(-2.5TA, -2.0TA)$

Company Status in 2015	Prediction Status in 2015 Operating	Bankrupt	Total
Operating	161	3	164
Bankrupt	5	9	14
Total	166	12	178

4 Discussions and Conclusions

From Table 1, the overall correctly prediction rate is 0.95 (169 out of 178). There were two types of errors: (i) the false positive rate that companies were predicted operational but, in fact, bankruptcy in 2015 is 0.036, and (ii) the false negative rate that companies were predicted to go bankrupt but remained operating in 2015 is 0.27. Theoretically speaking, one should select the thresholds which minimize the two types of errors. This can only be achieved numerically by using data with a very large group of companies. Comparing Tables 1 with 2, the prediction results are quite similar despite of much lower thresholds in the later case. It seems the results are somewhat robust with respect to the thresholds.

Selecting the threshold for working capital has been studied by Fu (2009). It has been shown that, for a threshold of $-1.5TA$, the one-dimensional BCP model of working capital on business bankrupt outperformed both the multiple-discriminant analysis and logistic regression analysis. This is one of the reasons that we select $a = -2TA$ for cashflow and $c = -1.5TA$ for working capital as the two-dimensional thresholds domain. We expected that a two-dimensional BCP model of working capital and cashflow performs better than the method proposed by Fu (2009) which was only based on working capital. The above numerical results in Section 3 substantiate this claim. It is clear that the main reason of this proposed method being more efficient and accurate is due to the fact that the trend $\boldsymbol{\beta}$ of working capital and cashflow, two most predictive variables, is simultaneously incorporated into the stochastic model. One may use different statistical methods to estimate the linear trend $t\boldsymbol{\beta}$ and covariance matrix $\boldsymbol{\Sigma}$. Based on our experience, it will have minimal effect on the outcome.

In view of our Remark 1, the two-dimensional stochastic model considered here can certainly be extended to a $d-$dimensional, $d > 2$, stochastic model. However, for $d > 4$, the state space associated with the imbedded Markov chain is very large and computation could be an issue.

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